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A THEORETICAL STUDY OF NONISOTHERMAL FLOW AND HEAT TRANSFER IN VERTICAL TUBES FOR FLUIDS WITH VARIABLE PHYSICAL PROPERTIES

A THESIS

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A THEORETICAL STUDY OF NONISOTHERMAL FLOW AND HEAT TRANSFER IN VERTICAL TUBES FOR FLUIDS WITH VARIABLE PHYSICAL PROPERTIES

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SUMMARY

An analytical study was made of laminar flow in the thermal entrance region of a vertical circular tube with uniform wall temperature. The most important assumptions were that the velocity profile at the entrance to the heated section was parabolic, that the temperature was uniform at this point, and that the radial velocity was negligible. The physical properties (viscosity, density, thermal conductivity, and heat capacity) were allowed to vary in a realistic manner. The objectives of the study were, first, to develop a mathematical model describing the physical situation; second, to find a method of solution for this model; and, finally, to present, in useable form, numerical results useful in design calculations and related work.

The general differential equations of continuity, motion, and energy were reduced, using the assumptions above and a boundary layer analysis, to a simplified form which describes the physical situation. The resulting equations were then approximated by a numerical scheme. The energy equation, which is non-linear, was approximated by a finite difference scheme which circumvented the problem of non-linearity without resorting to iterative calculations. The scheme is implicit; i.e., it was necessary to solve a system of simple linear algebraic equations. The equation of motion reduced to two multiple integrals. Special numerical quadrature formulas were developed to evaluate these integrals to achieve a high degree of accuracy. The magnitudes of radial velocity components were estimated by integrating the differential equation of

continuity using the trapezoidal rule. The numerical scheme was implemented on a Burroughs 220 Data Processing System.

The results from the numerical solution (velocity and temperature profiles) were combined and operated upon by the computer to give mean temperatures, local Nusselt numbers, and friction factors.

To check the potential accuracy of the numerical scheme developed, a comparison was made between the classical Graetz solution (1) and the numerical scheme applied to the case of constant physical properties. The agreement was within one per cent over the range ir which the Graetz solution is valid. As a further test of the suitability of the scheme, a comparison was made between computed results and the experimental data of Martinelli and co-workers (9). The agreement was within ten per cent, usually within five per cent.

The final objective of the study was to present results, useable for design and related calculations, sufficient to cover most commonly encountered calculations. It was desired to give results for heating in upflow, heating in downflow, cooling in upflow, and cooling in downflow. To present these results in a useable form, it was necessary to find correlation parameters so that results for several types of common substances could be treated as functions of a minimum number of variables. By assuming that very approximate equations relating density and viscosity to temperature would be sufficient to find correlation parameters, it was found that the Peclet number divided by the dimensionless axial coordinate, Pe/z, the ratio of the entrance and wall viscosities, μ_o/μ_w, and a free-convection parameter, $Fc = 2(Re/Fr)(\rho_w - \rho_o)/(\rho_w + \rho_o)$ were adequate to correlate the results if the heat capacity and thermal

conductivity were constant. It was then assumed that the latter two physical properties could be taken into account by evaluating Pe at a suitable temperature.

These assumptions proved correct: it was found that results for water, an oil, air, and helium were indeed functions of μ_o/μ_w, Fc, and Pe/z only. It proved necessary, however, to treat liquids and gases separately. For the local Nusselt number, the proper temperature at which to evaluate Pe proved to be the mean temperature. For correlating the mean temperature itself, it was likewise found necessary to evaluate Pe at the mean temperature. The wall temperature was found to be the correlating temperature for the friction factor. The local temperature, i.e., the temperature at the point in question, was found to be the proper temperature for correlating velocity and temperature profiles.

It was found that, using these temperatures, the results for the oil and water correlated closely under all conditions. This is felt to be especially significant, because the physical properties of the two substances are quite different; further, they vary in distinctly different ways with temperature. In general, the results for gases do not correlate well with those for liquids--probably because the density variation with temperature for gases is so different from that for liquids that free-convection forces are significantly affected. For small temperature differences, the results for gases and liquids do agree--and in this case, the density vs. temperature curve for gases can be considered to be almost linear, as is the case for liquids. Also, the results for liquids and gases were found to agree when free-convection

effects are small. The results for two relatively dissimilar gases, air and helium, were found to correlate well under all conditions.

Because the correlation methods appear to be satisfactory, the final objective of the study could be achieved: results over the range of most practical interest were computed and are presented in tables and graphs as functions of the pertinent correlating parameters. For liquids, viscosity ratios, μ_o/μ_w, of 0.05, 0.1, 0.2, 0.5, 1, 2, 5, 10, and 20 were considered; for gases, ratios of 0.5, 0.667, 1, 1.5, and 2 were investigated. Free-convection parameters, Fc, of -200, -100, -0.1, 0.1, 100, 200, 400, and 800 were studied for each viscosity ratio. Results are given in the range $10 \leq Pe/z \leq 10^4$, since this seems to be the range of most practical importance.

For the oil, because of its large Prandtl number, it was found that a significant change in the initial velocity profile (e.g. parabolic to flat) caused negligible change in the temperature profile. This occurs because the larger the Prandtl number, the faster the velocity profile develops compared to the temperature profile. This means that the results for the oil are essentially independent of the initial velocity profile shape. Thus, the results are applicable to flows with initially flat profiles as well as developed profiles. For water, with a Prandtl number of about five at room temperature, it is inexact to assume that the results are independent of the initial velocity profile shape--but the approximation is not too bad in many cases. For gases, the results must be applied only to cases in which the initial profile is parabolic.

In cases where Fc is less than 10.0 (free convection negligible), the results of this study may be applied equally well to horizontal tubes and vertical tubes.

The results of this study suggest that further experimental work is required, particularly in regions where flow becomes unstable (undergoes transition to turbulent flow). Further analytical work is needed, particularly including consideration of developing velocity profiles and of flow in horizontal tubes in cases where free-convection effects are important.

NOMENCLATURE

In this tabulation, dimensions are given in terms of mass (M'), length (L'), time (t'), and temperature (T'). Dimensional quantities are usually (but not always) primed in this study; non-dimensional quantities are not primed. Symbols used in a purely mathematical sense are not listed.

Symbol	Definition	Units
$\vec{A}$	area (vector quantity)	L'^2
A	area (scalar quantity)	L'^2
b	coefficient in viscosity-temperature function	$1/T'$
B_i	$-2 - (Re\rho_{in}c_{p_{in}}u_{in}(\Delta r)^2)/k_{in}\,\Delta z$	dimensionless
c_p	heat capacity, c'_p/c'_{p_o}	dimensionless
c'_p	heat capacity	L'^2/t'^2T'
c'_{p_A}	heat capacity evaluated at $1/2\,(T'_m + T'_o)$	L'^2/t'^2T'
c'_{p_m}	heat capacity evaluated at T'_m	L'^2/t'^2T'
c'_{p_L}	heat capacity evaluated at local temperature	L'^2/t'^2T'
c'_{p_o}	heat capacity evaluated at temperature T'_o	L'^2/t'^2T'
C.V.	control volume	---
D	tube diameter	L'
D_i	$-T_{i-1,n-1} - T_{i+1,n-1} + (B_i+4)T_{i,n-1} - \Delta r(T_{i+1,n} - T_{i-1,n})(1/r_i + (k_{i+1,n} - k_{i-1,n})/2k_{in}\,\Delta r)$	dimensionless

Symbol	Definition	Units
f	friction factor	dimensionless
f_w	hypothetical friction factor resulting if fluid were flowing isothermally at wall temperature	dimensionless
Fc	free-convection parameter, $2(Re/Fr)(\rho_w-\rho_o)/(\rho_w+\rho_o)$	dimensionless
Fr	Froude number, $V^2/g_z D$	dimensionless
g	magnitude of acceleration due to gravity	L'/t'^2
g_z	acceleration due to gravity in z-direction	L'/t'^2
G	pressure gradient, dp/dz	dimensionless
G_f	pressure gradient due to fluid friction	dimensionless
G_g	pressure gradient due to gravity	dimensionless
Gr	Grashof number, $D^3\rho_o' g_z \beta(T_w'-T_o')/\mu_o'^2$	dimensionless
Gr_w	Grashof number evaluated at wall, $D^3\rho_o' g_z \beta(T_w'-T_o')/\mu_w'$	dimensionless
h	local heat transfer coefficient, based on wall-to-mean temperature difference	M'/t'^3T'
h_{am}	average heat transfer coefficient, based on arithmetic mean temperature difference	M'/t'^3T'
h_m	average heat transfer coefficient, based on arbitrary mean temperature difference	M'/t'^3T'

Symbol	Definition	Units
H	enthalpy, $H'/c'_{p_o}T'_o$	dimensionless
H'	enthalpy	$M'L'^2/t'^2$
k	thermal conductivity, k'/k'_o	dimensionless
k'	thermal conductivity	$M'L'/t'^3T'$
k'_A	thermal conductivity evaluated at $1/2\ (T'_o + T'_m)$	$M'L'/t'^3T'$
k'_L	thermal conductivity evaluated at local temperature	$M'L'/t'^3T'$
k'_m	thermal conductivity evaluated at T'_m	$M'L'/t'^3T'$
k'_o	thermal conductivity evaluated at T'_o	$M'L'/t'^3T'$
k'_w	thermal conductivity evaluated at T'_w	$M'L'/t'^3T'$
L'	length	L'
M'	mass	M'
N	number of increments into which tube radius is divided; i.e., $N = R/\Delta r$	dimensionless
Nu	local Nusselt number, hD/k'_m	dimensionless
Nu_{am}	Nusselt number based on arithmetic mean temperature difference, $h_{am}D/k'_A$	dimensionless
$O(x)$	of order no greater than x	---
p	pressure, p'/ρ'_oV^2	dimensionless
p'	pressure	$M'/L't'^2$
$\Delta p'$	pressure drop	$M'/L't'^2$
$(\Delta p')_w$	hypothetical pressure drop resulting if fluid were flowing isothermally at wall temperature	$M'/L't'^2$
Pe	Peclet number, $DV\rho'_oc'_{p_o}/k'_o$	

Symbol	Definition	Units
Pe_A	Peclet number evaluated at 1/2 $(T'_o + T'_m)$, $DV\rho'_o c'_{p_A}/k'_A$	dimensionless
Pe_L	Peclet number evaluated at temperature prevailing at z and r being considered, $DV\rho'_o c'_{p_L}/k'_L$	dimensionless
Pe_m	Peclet number evaluated at T'_m, $DV\rho'_o c'_{p_m}/k'_m$	dimensionless
Pr	Prandtl number, $c'_{p_o}\mu'_o/k'_o$	dimensionless
P1	$\int_0^r \rho r dr$	dimensionless
P2	$\int_r^{1/2} (r/\mu)dr$	dimensionless
P3	$\int_r^{1/2} (P1/r\mu)dr$	dimensionless
q_{exp}	experimentally determined rate of heat transfer from wall to fluid	$M'L'^2/t'^3$
q_{calc}	predicted rate of heat transfer from wall to fluid	$M'L'^2/t'^3$
Q	rate of heat transfer from tube wall to fluid	$M'L'^2/t'^3$
r	radial coordinate, r'/D	dimensionless
r'	radial coordinate, measured from center of tube	L'
Δr	increment of length in radial direction	dimensionless
R	tube radius	L'
Re	Reynolds number, $DV\rho'_o/\mu'_o$	dimensionless

Symbol	Definition	Units
Re_w	Reynolds number evaluated at wall, $DV\rho_o'/\mu_w'$	dimensionless
t'	time	t'
T	temperature, $(T'-T_o')/(T_w'-T_o')$	dimensionless
T_{mv}	mean temperature corrected for effect of viscosity	dimensionless
$\overline{T}$	temperature, T'/T_o'	dimensionless
T'	temperature	T'
T_o'	initial temperature at entrance to heating section of tube	T'
T_m'	mean temperature	T'
T_w'	temperature at tube wall	T'
$(\Delta T')_m$	mean temperature difference	T'
u	axial velocity, u'/V	dimensionless
u'	axial velocity	L'/t'
v	radial velocity, v'/V	dimensionless
v'	radial velocity	L'/t'
$\overline{v}$	ρvr	dimensionless
V	average velocity at entrance to heating section of tube	L'/t'
$\vec{\overline{V}}$	total velocity (vector quantity)	L'/t'
w	rate of fluid flow	M'/t'
z	axial coordinate, z'/D	dimensionless
z'	axial coordinate, measured from start of heating section of tube	L'

Symbol	Definition	Units
Δz	increment of length in axial direction	dimensionless
$(\Delta z)_o$	initial increment of length in axial direction	dimensionless
α	correction factor for effect of viscosity on free convection	dimensionless
β	coefficient of thermal expansion of fluid	$1/T'$
γ	ratio of specific heats	dimensionless
ϵ	quantity small compared to unity	dimensionless
θ	correction factor for effect of viscosity on free convection	dimensionless
Φ	correction factor for effect of free convection on mean temperature	dimensionless
μ	viscosity, μ'/μ_o'	dimensionless
μ'	viscosity	$M'/L't'$
μ_o'	viscosity evaluated at T_o'	$M'/L't'$
μ_w'	viscosity at tube wall	$M'/L't'$
ρ	density, ρ'/ρ_o'	dimensionless
ρ'	density	M'/L'^3
ρ_m'	density evaluated at T_m'	M'/L'^3
ρ_o'	density evaluated at T_o'	M'/L'^3
ρ_w'	density at tube wall	M'/L'^3
τ	shear stress in fluid	M'/t'^2L'

CHAPTER I

INTRODUCTION

A basic understanding of problems of heat transfer and fluid friction in laminar flow inside tubes has been sought by numerous investigators for many years. Several problems of interest have been solved; many more have not. In particular, there has long been a need for an extensive study of non-isothermal flow in which the effect of temperature on the physical properties of the flowing fluid is taken into account. The results of such a study would be of immediate practical value in industrial design, in suggesting further experiments, and in contributing to the general fund of basic knowledge.

This study was an attempt to examine analytically one of the many important problems in this area: laminar flow in the thermal entrance region of a vertical circular tube with uniform wall temperature. The special feature of this study was that all the physical properties of the flowing fluid--density, viscosity, thermal conductivity, and heat capacity--were considered to be functions of temperature in a realistic manner.

The objectives of this study were: (1) to develop a mathematical model to describe the physical situation, basing this model on the general differential equations of continuity, motion, and energy; (2) to solve this mathematical model using numerical techniques; and (3), to obtain results for a large number of cases of practical interest

and to correlate these results in a useable fashion for design calculations.

As mentioned above, a number of important solutions for special cases of heat transfer from a circular tube to a fluid in laminar flow have been presented in the literature. An analytical solution was first reported by Graetz (1) for the case of constant physical properties, constant tube wall temperature, and fully developed velocity profile. With improvements and extensions (2,3) which have later appeared, this problem appears to have been completely solved.

Important variations of this problem have likewise been considered: constant wall heat flux (4), transient conditions (5,6), and frictional heating (7). The effect of developing velocity profile was considered and solved by Kays (8).

For many practically important flows, however, the assumption of constant physical properties is not adequate--under conditions in which there are large variations in physical properties, these idealized solutions do not agree with experiment (9). Accordingly, later investigators began to study the effect of variable properties (10,11,12,13,14, 15,16,17). The more significant of these attempts are discussed below.

Cherry (17) and Yamagata (10) studied the effect of variable viscosity on pressure drop and heat transfer in the thermal entrance region by assuming that, as a first approximation, the temperature profiles given by isothermal solutions are correct. From these temperature profiles, the velocity profiles were computed. The major limitations of this work are that free-convection effects were not considered and that there is no really satisfactory method of determining how good

the approximation to the temperature profile is.

A later study by Martinelli and Boelter (15), based largely on physical reasoning, included only the effect of free convection. These authors derived the form of a semi-theoretical equation which predicted that the arithmetic mean Nusselt number would depend on the groups Pe_A/z and Gr_w/Re_w in a prescribed fashion. When applied to experimental data (9), however, the model was only partially successful, even when one constant was empirically adjusted.

Pigford (11) realized that realistic attempts to theoretically describe laminar flow in a heated vertical tube must take into account both the variations of density and viscosity with temperature. Pigford assumed that the temperature field may be described by Leveque's solution of the energy equation (18). He then showed that if it is assumed that

$$\rho' = \rho'_o \left[1 - \beta(T' - T'_o)\right] \tag{1}$$

$$\frac{1}{\mu'} = \frac{1}{\mu'_o} \left[1 + b(T' - T'_o)\right] \tag{2}$$

the resulting conditions can be described by the groups μ_o/μ_w, Pe/z, and Gr_w/Re_w.

Rosen and Hanratty (16) discussed Pigford's solution and attempted to present an improved solution of the integral type, assuming the same functions as Pigford for density and viscosity, a developed velocity profile, but retaining the inertia terms in the equation of motion, which Pigford did not do. Unfortunately, the agreement between their calculations and experimental Nusselt numbers is not satisfactory.

Scheele et al. (19) report some valuable experimental observations. They found that, in experiments on the heating and cooling of water in vertical tubes, distortions in the velocity profiles caused the flow to become turbulent at Reynolds numbers far less than in isothermal flow. In particular, they found, in visual experiments, that when cooling in upflow, transition appeared soon after the velocity gradient at the wall became zero and a reversal of flow occurred. For heating in upflow, instabilities were always observed downstream of the point where the center-line velocity dropped to zero.

Martinelli and co-workers (9) presented a vast amount of experimental data for heating an oil and water, including both heat transfer and pressure drop data. Watzinger and Johnson (20) measured velocity profiles in cooling water.

Koppel and Smith (12) presented a numerical solution for laminar flow heat transfer with variable physical properties. They analytically examined the behavior of carbon dioxide near the critical point, where the physical properties change radically with temperature. Their method, while probably quite adequate for their investigations, cannot be used for the wide variety of conditions considered in this study because of their assumption that the product of axial velocity and density is a function of radius only. Despite the fact that continuity indicates that the rate of change of axial velocity with axial position is small does not imply that, over a sufficient distance, the total change is insignificant. A solution which attempts to be compatible with experimentally measured velocity profiles must allow for such a change. Proof of this statement is evident upon examination of the profiles measured by Watzinger and Johnson (20).

CHAPTER II

DEVELOPMENT OF EQUATIONS AND METHOD OF SOLUTION

In this chapter, the problem being solved is explained in detail. The general differential equations of continuity, momentum, and energy are first reduced to simplified forms which, along with the pertinent boundary conditions, describe the physical situation of interest. These equations are then replaced by finite difference approximations, suitable for solution on a high-speed digital computer. This chapter is of a summary nature with most of the details in Appendices A, B, C, D, and E.

Statement of Problem.--The type of flow considered here is the steady laminar flow of a Newtonian fluid, either upward or downward, in a vertical tube. At the entrance to the heating section of the tube, the velocity profile is considered to be parabolic. Further changes in the velocity profile are assumed to be due to physical properties variations and not to radial velocity components, which are assumed to be small. The flow is assumed to be axially symmetric, the angular velocity is assumed to be negligible, and the Mach number is assumed to be much less than unity. The bulk viscosity is assumed to be zero, and the Peclet number is assumed to be greater than about ten.

The fluid is assumed to be initially at a uniform temperature T'_o. Starting at the entrance to the heating section, the tube wall temperature is maintained at a constant temperature T'_w over the entire heated length.

Development of Equations.--Under the assumptions above, the general equations given by Bird, Stewart, and Lightfoot (21) are shown in Appendix A to reduce to the non-dimensional equations (Motion and Continuity)

$$G = \frac{\frac{Re}{Fr}\left\{\int_o^{1/2} \rho r \left[\int_r^{1/2} \frac{1}{y\mu}\left(\int_o^y s\rho ds\right)dy\right] dr\right\} - \frac{1}{8}}{\frac{Re}{2}\int_o^{1/2} \rho r \left(\int_r^{1/2} \frac{y}{\mu}\, dy\right) dr} \qquad (3)$$

(Motion)

$$u = \frac{Re}{Fr}\int_r^{1/2} \frac{1}{\mu r}\left(\int_o^r y\rho dy\right) dr - \frac{ReG}{2}\int_r^{1/2} \frac{r}{\mu}\, dr \qquad (4)$$

(Energy)

$$\rho c_p u \frac{\partial T}{\partial z} = \frac{1}{rPe}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) \qquad (5)$$

The boundary conditions associated with the energy equation, (5), are

$$r = 1/2\ ,\ T = 1 \qquad (6a)$$

$$r = 0,\ \frac{\partial T}{\partial r} = 0 \qquad (6b)$$

and

$$z = 0,\ T = 0 \qquad (7)$$

The symbols used in these equations are given in the Nomenclature section.

Numerical Scheme.--To solve the system of equations (3), (4), and (5), suitable numerical methods were developed. It will be noted that the energy equation is non-linear since ρ, c_p, and k are functions of temperature, and that the equations are coupled in that the temperature

field is dependent on the velocity field and vice versa.

It was decided to develop a numerical scheme which would circumvent the problems of non-linearity and interaction between equations. Further, the desired scheme had to be stable and possess sufficient accuracy per unit computational effort that useable results could be obtained without excessive use of computer time. Such a scheme is developed in Appendix B. The scheme is so algebraically complicated that there could be little gain in clarity in reproducing it in this section.

The numerical scheme developed was programmed for a Burroughs 220 Data Processing System. A discussion of the program, including a flow sheet and the program itself, is given in Appendix C.

Application of the program to a specific substance requires that the physical properties of the substance (ρ', c_p', k', and μ') be represented by reasonably accurate empirical equations. The physical property data and corresponding empirical equations for the substances considered in this analysis are given in Appendix E.

CHAPTER III

DEVELOPMENT OF CORRELATION SCHEME

A major objective of this study was to develop a detailed numerical scheme for solving the problem for any given real material, but it was also considered quite important to present solutions for as many cases as possible as compactly as possible. This means that it was necessary to find, if possible, practical, semi-empirical correlations of the results for substances of varying physical properties. In this chapter, a discussion of determining suitable parameters for correlating T_m, Nu, f/f_w, velocity profiles, and temperature profiles for different substances is given.

At first glance, this task might appear hopeless--viscosity, density, thermal conductivity, and heat capacity vary from substance to substance in different ways. Further, simple expressions such as those used by Pigford (11) for density and viscosity variation,

$$\rho' = \rho'_o \left[1 - \beta(T' - T'_o)\right] \tag{8}$$

$$\frac{1}{\mu'} = \frac{1}{\mu_o} \left[1 + b\,(T' - T'_o)\right] \tag{9}$$

are not adequate over wide ranges of temperatures for many substances of interest. Still, these approximations are descriptive of general trends in physical property variation.

By substituting equations (8) and (9) into his theoretical

equations describing the physical situation and non-dimensionalizing, Pigford showed that the solution to his problem was a function of μ_o/μ_w, Pe/z, and Gr_w/Re_w only. Suppose we assume that μ_o/μ_w, Pe/z, and $Fc = (Re/Fr)\beta(T'_w - T'_o) = (Gr_w/Re_w)(\mu_o/\mu_w)$ are the controlling parameters for the more general problem considered here, even though equations (8) and (9) by no means fit the data well. This correlation procedure might be expected to succeed if the results do not depend critically on the parameters μ_o/μ_w and Fc (i.e., if a 10-20 per cent change in the values of these parameters changes the results very little). At least one modification is necessary, however: in the Grashof number, the quantity $\beta(T'_w - T'_o)$ appears. If β is a constant, as assumed by Pigford, there is no further problem with this term. However, for substances such as air and water, β varies greatly with temperature. We note, however, that if

$$\rho' = \rho'_o \left[1 - \beta(T' - T'_o)\right] \tag{10}$$

then

$$\rho'_w = \rho'_o \left[1 - (T'_w - T'_o)\right] \tag{11}$$

or

$$\beta(T'_w - T'_o) = \frac{\rho'_o - \rho'_w}{\rho'_o} = \frac{\rho_o - \rho_w}{\rho_o} \tag{12}$$

It seems evident that this approximation can be improved by replacing $\beta(T'_w - T'_o)$ by $2(\rho_w - \rho_o)/(\rho_o + \rho_w)$. For liquids, this is virtually equivalent to the result (12). For gases, however, where the density change can be so large, the latter replacement appears to be more satisfactory. In fact, for gases, where $\rho \simeq 1/\overline{T}$,

$$(T'_w - T'_o)\beta\Big|_{\frac{(T'_w+T'_o)}{2}} = \frac{2(\rho_o - \rho_w)}{(\rho_o + \rho_w)} \qquad (13)$$

To give the sign of the free-convection parameter some meaning, it is defined to be $Fc = 2(Re/Fr)(\rho_w - \rho_o)/(\rho_o + \rho_w)$. For heating, $\rho_w - \rho_o$ is negative. For upflow, the Froude number, $V^2/g_z D$, is negative ($g_z = -g$); for downflow, it is positive. Thus, for heating in upflow and cooling in downflow, in which free-convection forces aid forced convection, Fc is positive. It is negative when free convection opposes forced convection.

Guided by these considerations the term $\beta(T'_w - T'_o)$ was replaced in the Grashof Number by the more general term $2(\rho_w - \rho_o)/(\rho_o + \rho_w)$.

To summarize, the controlling parameters were felt to be $Fc = 2(Re/Fr)(\rho_w - \rho_o)/(\rho_w + \rho_o)$, Pe/z, and μ_o/μ_w. It must be made clear that if the empirical equations for physical properties used in this study are substituted into equations (3), (4), and (5) the groups Fc, μ_o/μ_w, and Pe/z do not appear as theoretical correlating parameters; these groups were found by considering much simpler case. It was felt that groups found for the simple case were approximately applicable to the more complicated situation.

Still left unanswered is the question of the effect of variable heat capacity and thermal conductivity. This problem is ;treated in the next chapter.

CHAPTER IV

DISCUSSION OF RESULTS

In this chapter, the numerical scheme discussed in Chapter II and the correlation scheme discussed in Chapter III are confirmed. With these techniques thus verified, they are used to produce significant amounts of results for general design and research use with liquids and gases.

Comparison with Graetz Solution.--As a first check on the accuracy of the numerical scheme, the classical Graetz problem (discussed in detail by Jakob (18)) was solved using the numerical scheme, and the results were compared with the analytical solution of Graetz. Numerical values for the Graetz solution were computed using all the eigenvalues presently available, the eleven reported by Brown (2). A comparison between the Graetz and numerical solutions for Nu, Nu_{am}, and T_m is given in Table 1.

It is seen that below Pe/z = 1000, the agreement is quite good, suggesting strongly that the interval sizes chosen ($\Delta r = 0.025$, $(\Delta z)_o = 10^{-6}$ Pe) are quite adequate for describing the problem. Of particular interest is the fact that the local Nusselt number agrees quite well with the analytical value over most of the range of the solution. This lends a great deal more confidence to the solution because, in calculating the local Nusselt number, it is necessary to take a numerical derivative of the computed temperatures at the tube wall. The close agreement between numerical and analytical solutions thus means that

Table 1. Comparison of Graetz Solution and Numerical Solution.

	Mean Temperature		Arithmetic Mean Nusselt Number		Local Nusselt Number	
Pe/z	Numerical	Graetz	Numerical	Graetz	Numerical	Graetz
10000	0.0128	0.0164	32.2	41.3	28.2	16.5
6250	0.0177	0.0202	28.0	31.8	22.1	15.7
5000	0.0206	0.0226	26.0	28.6	19.8	15.2
2000	0.0377	0.0385	19.2	19.6	13.4	12.5
980	0.0600	0.0605	15.3	15.2	10.3	10.0
746	0.0715	0.0719	13.8	13.9	9.31	9.17
435	0.101	0.101	11.6	11.6	7.71	7.68
220	0.154	0.154	9.20	9.21	6.18	6.18
97.1	0.255	0.253	7.08	7.04	4.82	4.88
64.9	0.322	0.321	6.21	6.19	4.36	4.41
43.3	0.406	0.403	5.51	5.47	4.02	4.06
20.5	0.599	0.597	4.39	4.37	3.72	3.72
18.6	0.628	0.626	4.26	4.24	3.67	3.70
10.0	0.811	0.810	3.41	3.41	3.64	3.66

the numerical results must be quite accurate.

For $Pe/z \geq 1000$, the agreement is not so good; however, the inaccuracy seems to be more in the Graetz solution than in the numerical solution. Even with eleven eigenvalues, accuracy seems to be lost above $Pe/z = 1000$, as a plot of the values in Table 1 shows. The Leveque solution (18) is perhaps a closer approximation here, but its accuracy is questionable. For example, Jakob (18) reports that a very good fit to the Leveque solution is

$$T_m = 6.55\ (Pe/z)^{-2/3} \tag{14}$$

However, this approximation nowhere converges to the Graetz solution; the solutions simply do not meet. The numerical solution has about the same -2/3 slope on a log-log plot as the Leveque solution for Pe/z

up to 10,000; accordingly, the numerical solution may be accurate even in this region, $1000 \leq Pe/z \leq 10{,}000$. Since this cannot be proved, however, the results in this region must be regarded with less confidence than those in the range $10 \leq Pe/z \leq 1000$. A plot of the Graetz, Leveque, and numerical solutions is given in Figure 1.

It should be noted also that the agreement between the analytical and numerical solutions gives strong empirical support to the idea that the numerical scheme used is indeed stable and convergent.

Near $Pe/z = 10$, the local Nusselt number begins to oscillate. This oscillation does not indicate instability, for the actual solution (temperature profile) and the mean temperature behave normally. Rather, it is indicated that as the temperature profile becomes more highly developed, accuracy is lost in the calculation of the local Nusselt number, since differences of numbers of comparable size are involved. As discussed later, this problem is even more severe in cases where the temperature profile develops still faster. Accordingly, local Nusselt numbers must be regarded with suspicion for Pe/z near 10. Fortunately, this quantity is of little practical value in most applications.

Comparison with Experiment.--The next step in establishing confidence in the numerical scheme is comparison with experimental data. A large amount of data were reported by Martinelli and co-workers (9) for heating water and an oil in vertical tubes under conditions compatible with the assumptions made in this study. A summary comparison of experimental and computed results and the range of the experimental results is given in Table 2.

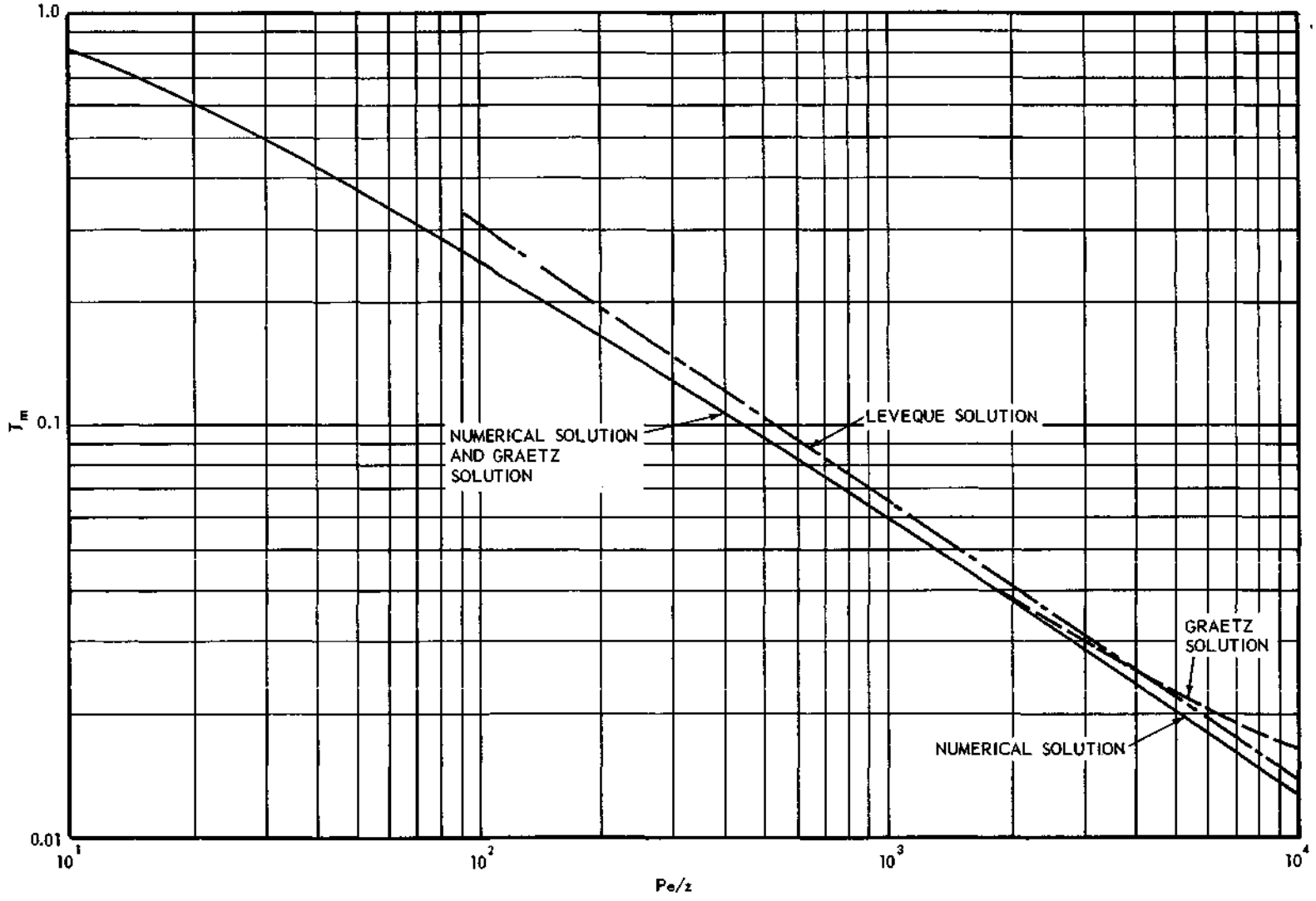

Figure 1. Comparison of Constant Physical Property Solutions.

Table 2. Comparison with Experiment.

Investigator	Run No.	Sub-stance	Re	Pr	z	μ_o/μ_w	Fc	q^*_{exp}, Btu/hr	q_{calc}, Btu/hr
Alves & Southwell	4	oil	516	680	602	17	1	15,860	15,240
Alves & Southwell	13	oil	76.7	711	602	18	5	6,890	6,710
Alves & Southwell	15	oil	7.33	697	602	18	62	1,937	1,977
Alves & Southwell	20	oil	33.9	741	602	19	12	5,250	4,840
Alves & Southwell	27	oil	2150	244	602	6	1	13,780	12,730
Alves & Southwell	4a	oil	516	680	297	17	1	10,370	10,310
Alves & Southwell	13a	oil	76.7	711	297	18	5	4,700	4,550
Alves & Southwell	15a	oil	7.33	697	297	18	62	1,852	1,680
Alves & Southwell	20a	oil	33.9	741	297	19	12	3,670	3,360
Alves & Southwell	27a	oil	2150	244	297	6	1	8,330	8,420
Weinberg	5a	oil	490	463	126	12	1	3,950	3,879
Weinberg	7a	oil	285	599	126	15	1	3,574	3,850
Weinberg	47	water	1769	6.33	126	3	314	7,500	7,200
Weinberg	48	water	1567	6.24	126	3	370	6,980	7,360

*Based on rate of steam condensation.

In this table, calculated rates of heat transfer are compared with those measured experimentally. The heat transfer rates and experimental conditions were reported by Martinelli (9); Re, Pr, μ_o/μ_w, Fc, and q_{calc} were calculated in this study. The discrepancies appear to be well within the experimental error generally experienced by these experimenters.

Another type of comparison is shown in Figure 2. Martinelli *et al.* reported a large number of runs with virtually the same viscosity ratio,

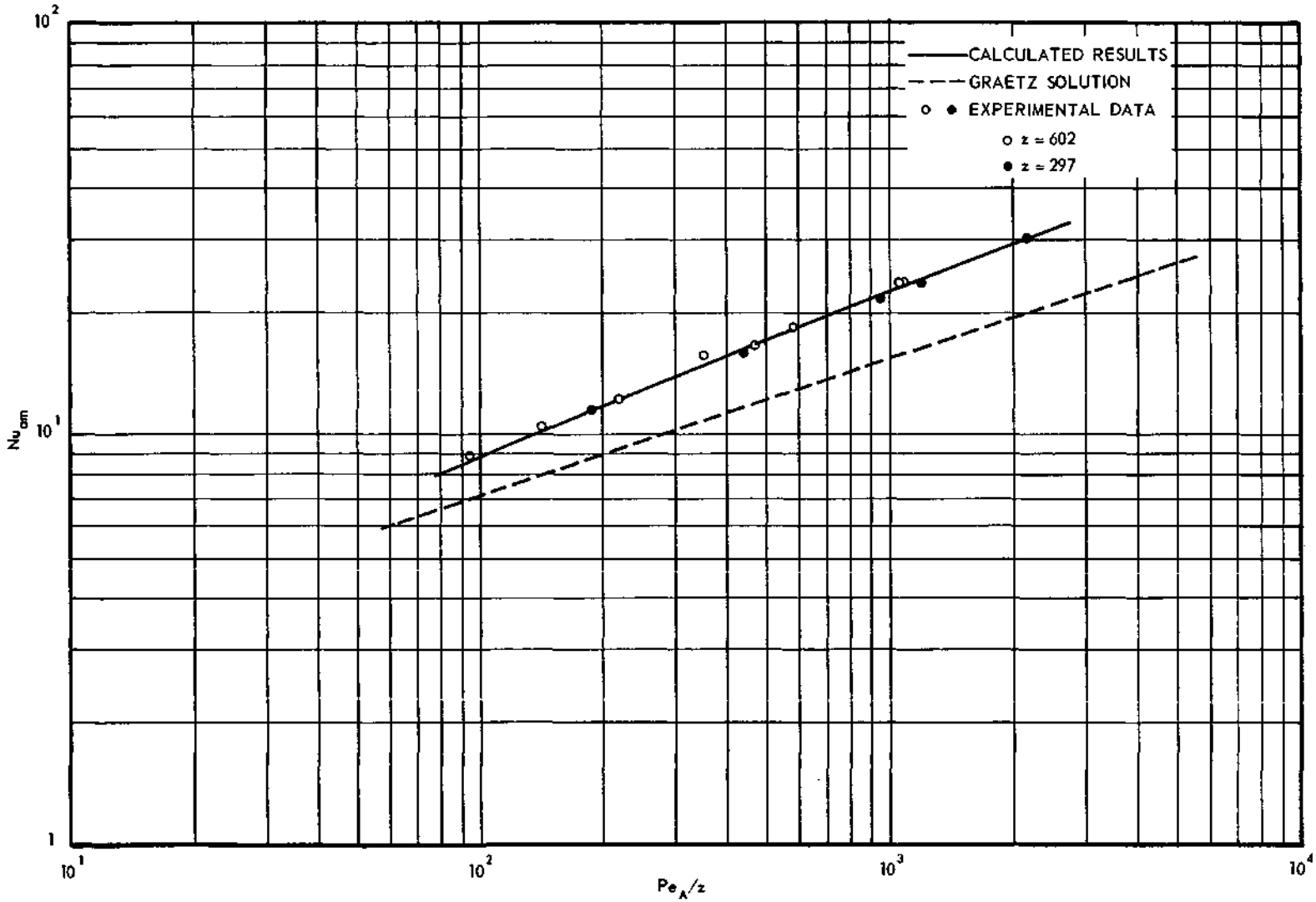

Figure 2. Comparison of Calculated and Experimental Nusselt Numbers.

μ_o/μ_w, and with free-convection parameters less than about five (which means that free-convection effects were negligible in these runs). Accordingly, it is an interesting confirmation of the theories in this study that these experimental arithmetic mean Nusselt numbers lie quite near the computed results using the conditions of only one of these runs (run 13 of Alves and Southwell). It is important to note that free convection is not important in these runs. The deviation of the results from the Graetz solution (which is plotted for comparison) is due almost entirely to the effect of temperature on viscosity. This emphasizes that the effect of viscosity variation can be significant.

It will be noted that only two of the many runs using water reported by Martinelli are listed in Table 2. This is because in all other reported water runs the flow became unstable before reaching the tube exit--unstable in the sense that the center-line velocity dropped below zero. In fact, in run 48, the center-line velocity dropped below zero before the tube exit was reached, but the results were extrapolated to the tube exit. Scheele et al. (19) report that instability always results downstream of the point where the center-line velocity drops to zero, but that this instability may be only incipient at this point--the flow may continue without transition for some length. Once any velocity drops below zero, however, the numerical scheme becomes unstable, as a glance at Appendix D will reveal, so calculations must be terminated at this point. Thus, there is some unknown length from the point where the flow shows a tendency to instability to the point at which the flow actually becomes unstable. It was assumed that, in run 48, the flow did not actually undergo transition to turbulence.

Martinelli (9) also reported pressure drop data which can be compared with the computed results. Unfortunately, the total pressure drop is not reported--only the pressure drop due to frictional forces is given. This is unfortunate because in several of the runs, the pressure drop due to gravity is quite large compared to the frictional pressure drop. To compute the frictional pressure drop, it was necessary to subtract the gravity head from the total head--two numbers of quite often comparable size. Further, the gravity head cannot be calculated with accuracy without knowing the mean temperature at each point in the tube. Unfortunately, this mean temperature was measured by Martinelli and co-workers at only two points, and had to be estimated at other points. Consequently, rather strange results were reported in some cases--for example, Alves and Southwell report a positive pressure gradient for run 9--a result probably due to incorrectly estimated values of mean temperature.

With these limitations in mind, a comparison between experimental and calculated pressure gradients is shown in Figure 3 for two runs in which the frictional pressure drop was of comparable magnitude with the gravity head. It will be noted that the agreement is fairly good--for lack of more information, the discrepancy can easily be attributed to the fact that the mean temperature was not known accurately at a sufficient number of points. It should be noticed that the experimental values are consistently lower than the calculated values--a fact indicating that perhaps the calculations by Martinelli and co-workers always gave results that were somewhat low. This conclusion is in agreement with the fact that run 9 indicated a positive pressure gradient up the tube.

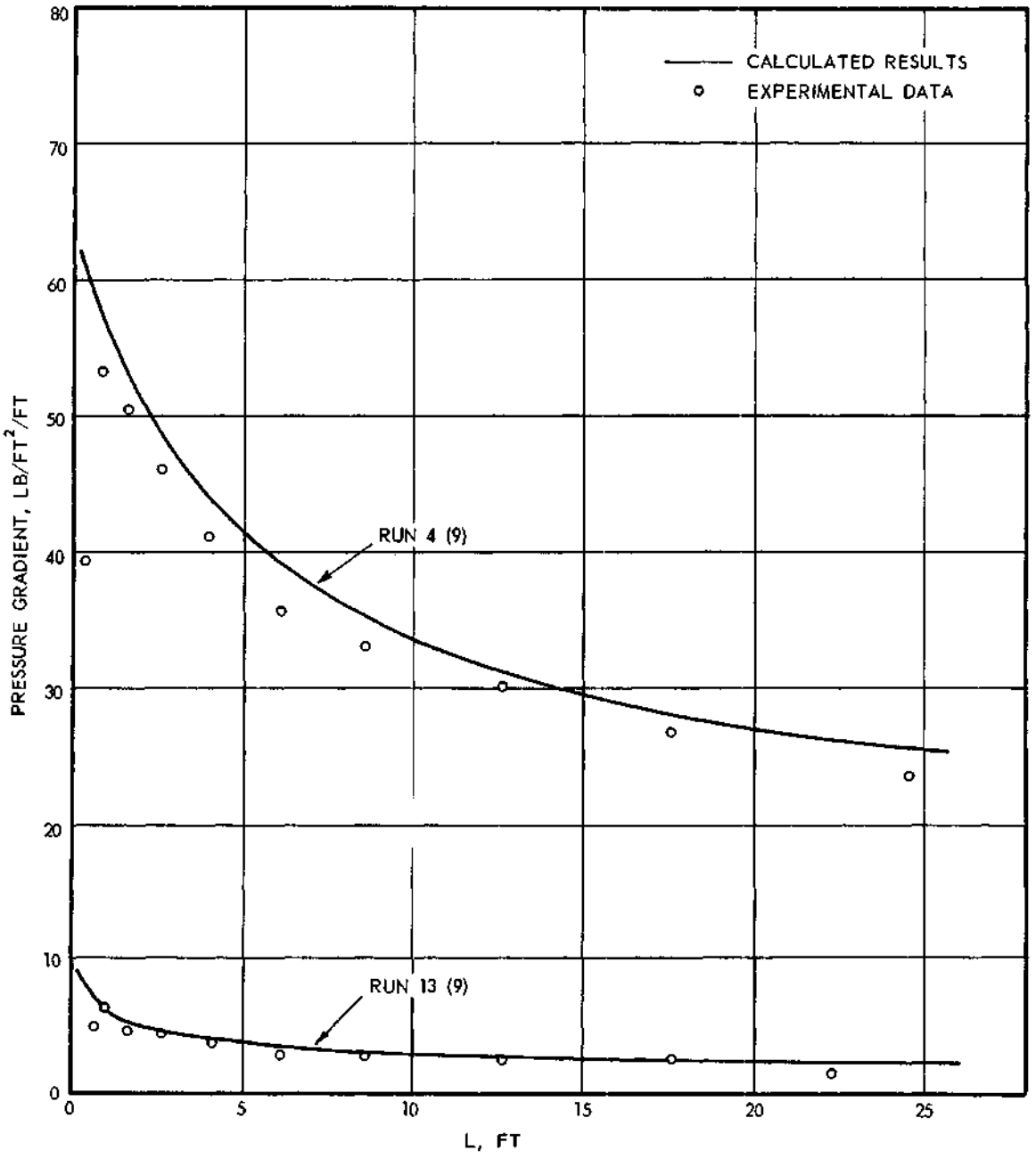

Figure 3. Comparison of Calculated and Experimental Pressure Drops.

Confirmation of Parameters for a Single Substance.--Since the numerical scheme has been shown to be satisfactory, it is now pertinent to seek the answer to another type of question: are the parameters developed in Chapter III adequate for correlating the important results of the numerical calculations? An answer to this question must be sought in two stages. First, it is necessary to consider a single substance flowing under varying conditions. Next, it is necessary to consider the results for various substances compared with one another. The first-stage testing of the correlation parameters was performed on water alone and "oil A" (9) alone. The physical properties of these substances are given in Appendix E.

Water was studied analytically under conditions such that free convection was important, since, for water, the density is very poorly represented by equation (8). Thus, this offers an excellent method of checking whether constancy of the free-convection parameter Fc is adequate for satisfactory correlation of results.

Oil A was studied analytically under conditions such that viscosity variation was important, since, for this substance, the viscosity is very poorly represented by equation (9). (The viscosities of the substances considered in this study are shown in Figure 34). Thus, study of oil A offers an excellent opportunity to see if constancy of the viscosity ratio, μ_o/μ_w, is sufficient for correlating results.

The conditions under which the calculations were made are given in Table 3. For a given substance, a wide range of parameters, such as Reynolds number and temperature range, were used, while μ_o/μ_w and Fc were held constant. All calculations were for heating of liquids in upflow.

Table 3. Summary of Calculations for Confirmation of Parameters--Upflow of Liquids.

Substance	Re	Pr	T'_o, °R	T'_w, °R	1/Fr	μ_o/μ_w	Fc
Water	10.0	5.89	540.0	640.4	-754.8	2.5	200
Water	100.0	3.93	573.0	699.1	- 45.2	2.5	200
Water	1000.0	4.78	556.0	667.8	- 5.83	2.5	200
Oil A	10.0	800	540.0	602.6	- 0.392	5.0	0.1
Oil A	100.0	384	565.0	649.2	- 0.0308	5.0	0.1
Oil A	1000.0	224	589.0	700.3	- 0.0022	5.0	0.1

Values of Nu, T_m, and f/f_w were compared at constant values of Pe/z, evaluated at the proper correlating temperature. (In plotting these results, it was necessary to anticipate the correct correlation temperatures, which are discussed more fully in the next section.) Some of these comparisons are shown in Figures 4 through 7. It is seen in these figures that, at least for single substances, the choice of correlation parameters seems to be quite adequate.

In a sense, it is fair to say that the results are correlated for "different" substances because of the different temperature ranges considered. For example, in the calculations with oil A at a Reynolds number of 10, the viscosity variation between the entrance and wall temperatures is quite different from that with Re = 1000, because the temperature range is significantly different.

Confirmation of Parameters for Liquids and Gases.--Even though the excellent correlation of results obtained in looking at substances one at a

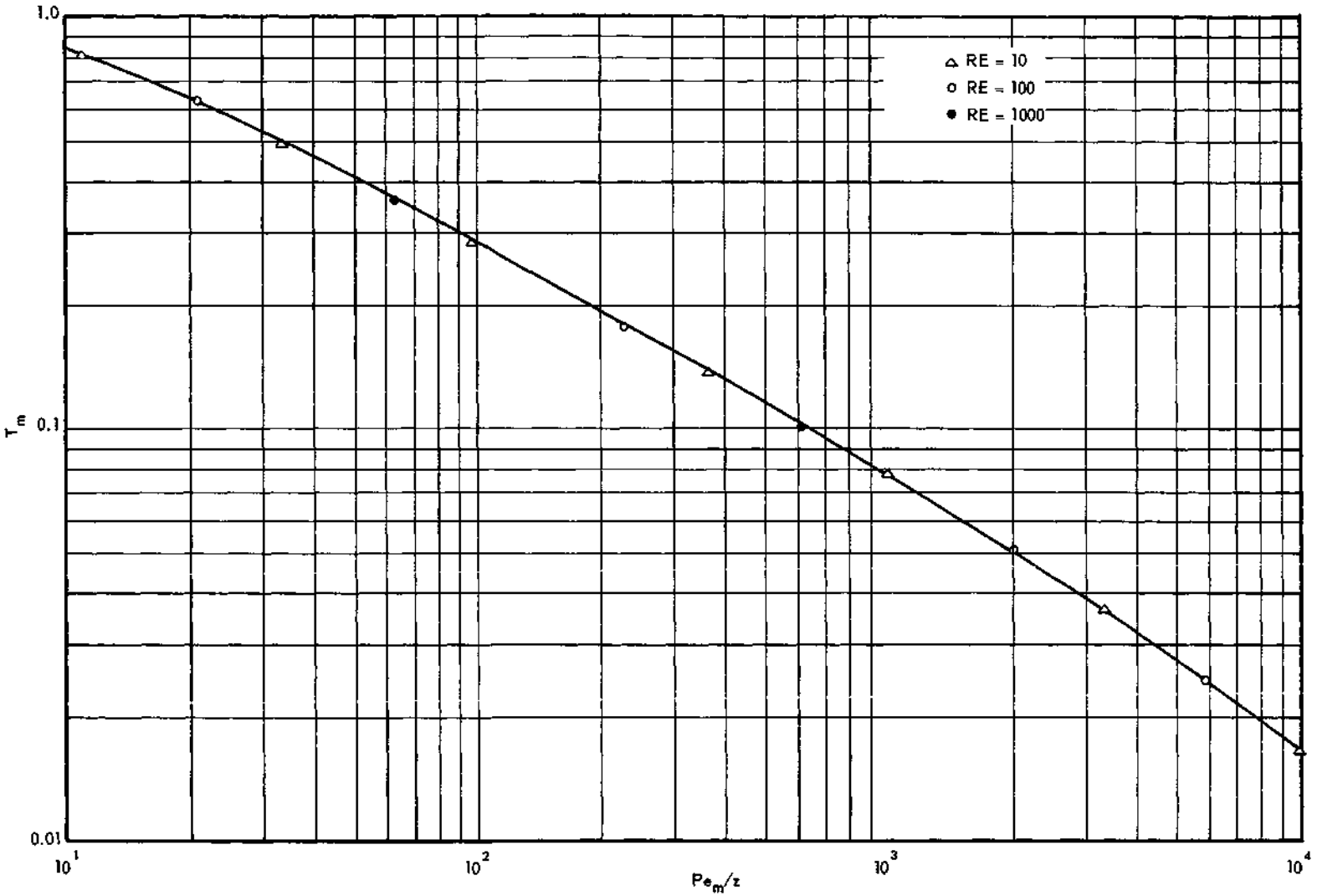

Figure 4. Comparison of Mean Temperatures for Oil A.

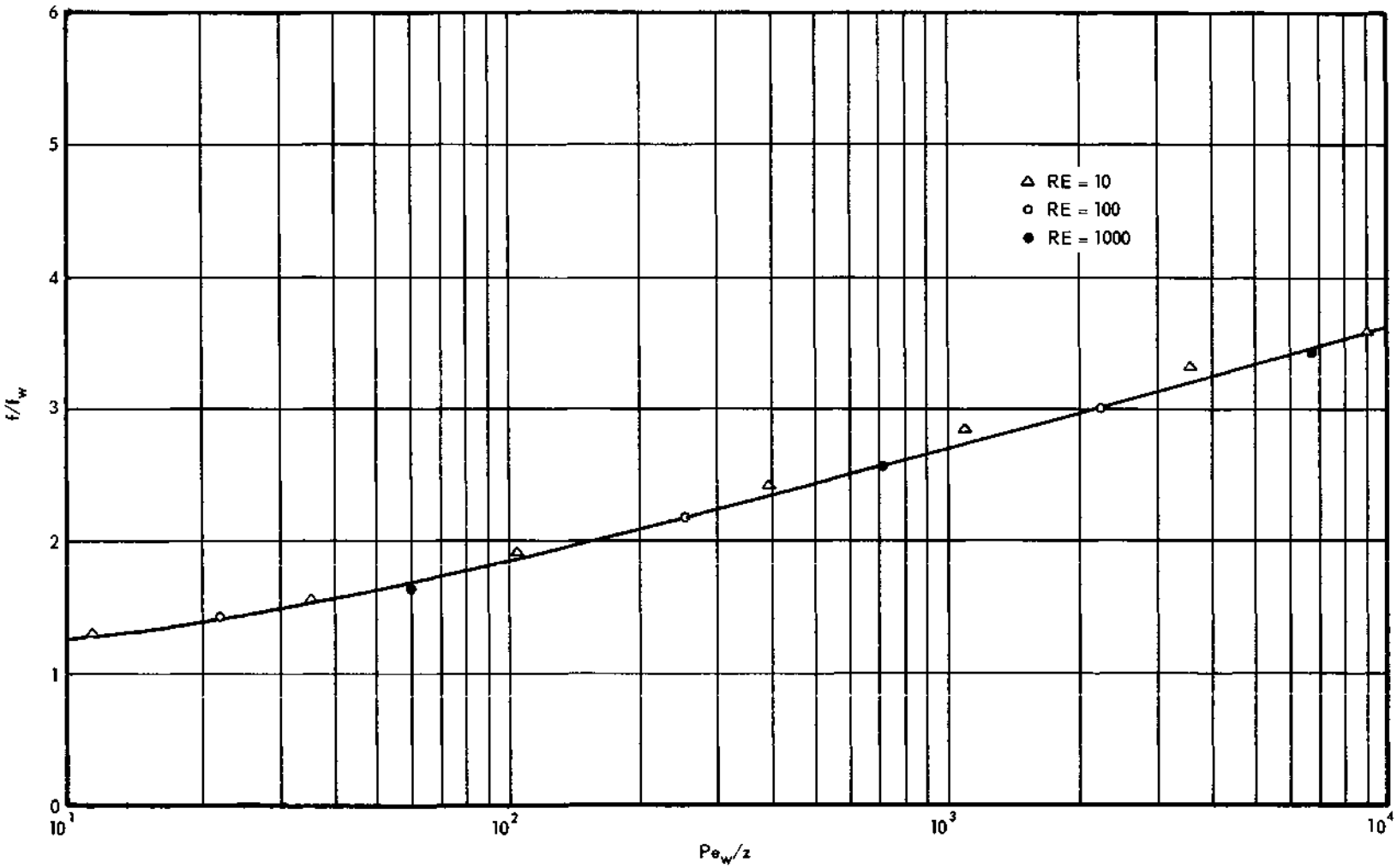

Figure 5. Comparison of Friction Factors for Oil A.

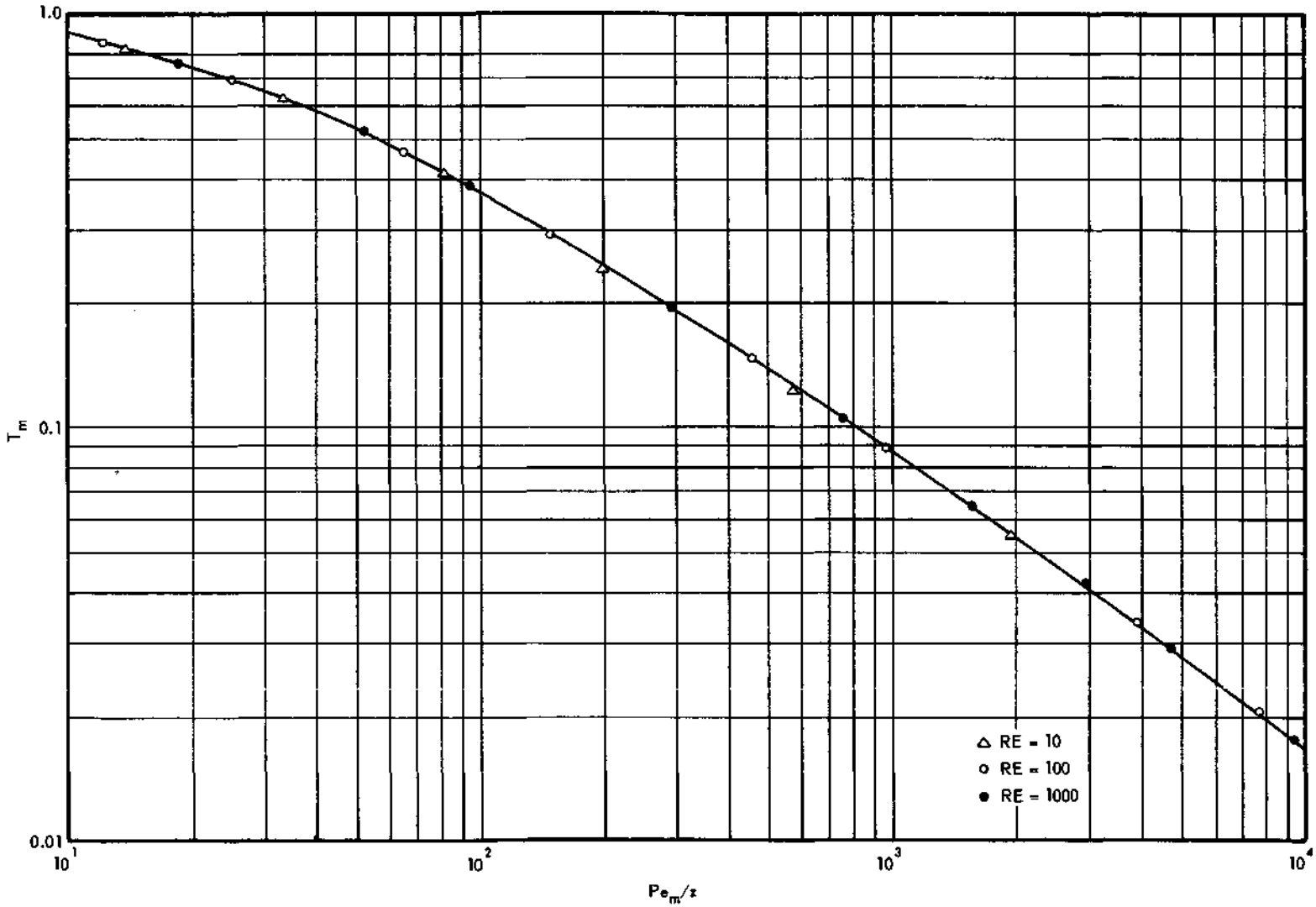

Figure 6. Comparison of Mean Temperatures for Water.

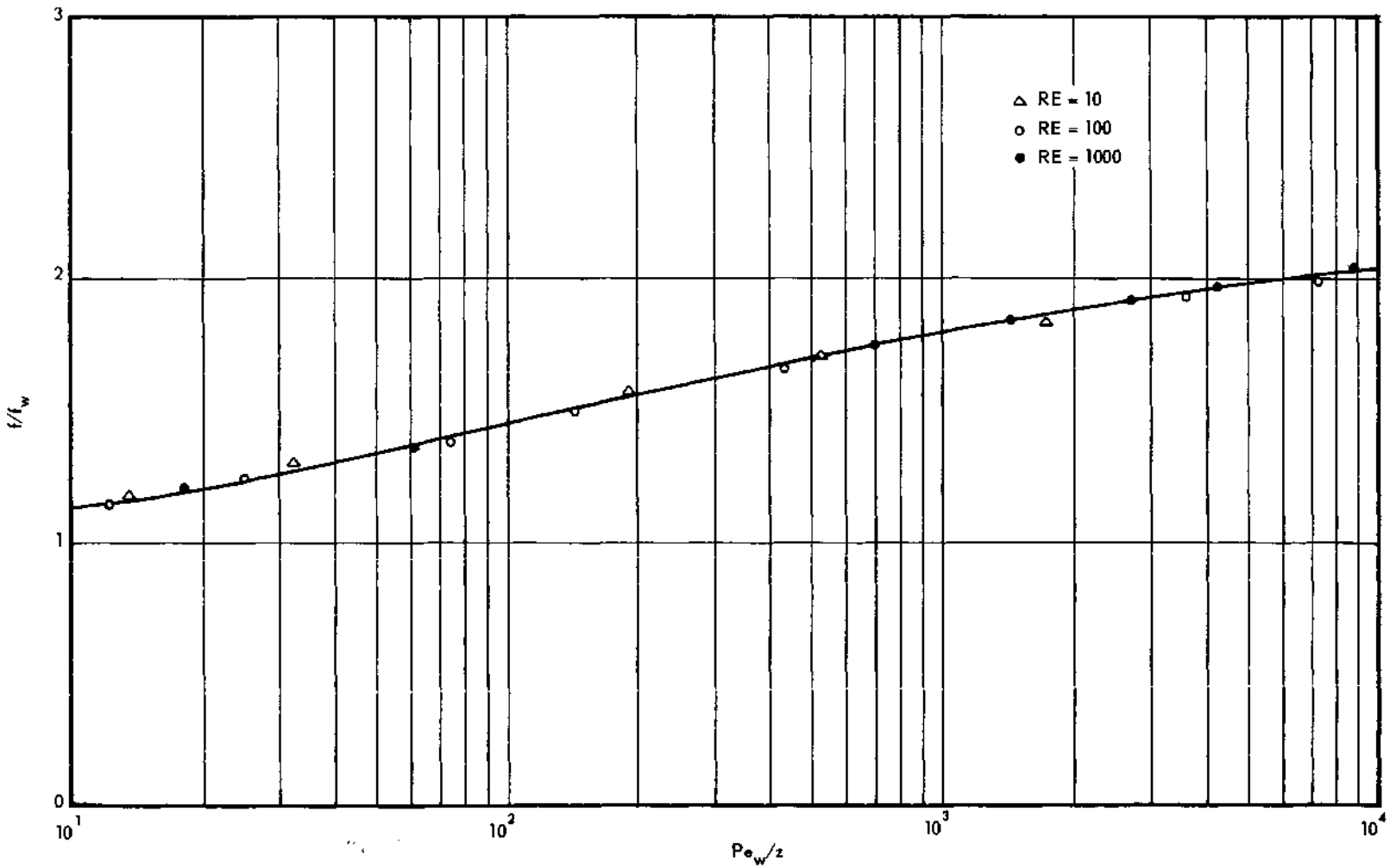

Figure 7. Comparison of Friction Factors for Water.

time is encouraging, it is still prudent to see if results for different substances correlate. Further, it is necessary to consider both heating and cooling.

The liquids water and oil A were chosen to be compared for several reasons. For oil A, the density is a linear function of temperature; for water, the density can be adequately represented only by a second degree polynomial. (The physical properties of these substances are given in Appendix E.) The heat capacity of water is essentially independent of temperature, while the heat capacity of oil A increases by almost 20 per cent with a 150° F increase in temperature. The thermal conductivity of oil A decreases slightly with increasing temperature. For water, the thermal conductivity increases by over 10 per cent with a 140° F increase in temperature. Finally, the effect of temperature on the viscosity of oil A is significantly greater than that for water. Thus, because these substances are so dissimilar, a comparison of them gives real credence to the proposed correlation.

Conditions for the comparisons between oil A and water are summarized in Table 4. Sample results, typical of those for other conditions, are shown in Figures 8 through 16. Examination of these figures indicates that the correlation scheme is quite excellent, except that values of velocity appear to differ by as much as 10 per cent in some extreme cases; in other cases, however, velocity profiles correlate quite well.

The effect of variable k and c_p was found to be satisfactorily taken into account by evaluating Pe at T_m for correlating Nu and T_m itself, and at T_w for correlating the friction factor ratio, f/f_w.

Table 4. Conditions for Comparison Calculations.

Substance	μ_o/μ_w	Fc	T'_o, °R	T'_w, °R	Re	1/Fr	Pr
Oil A	3.4	0.1	610	700	26.3	-0.105	153
Oil A	3.4	100	610	700	837	-3.30	153
Oil A	3.4	200	610	700	1182	-4.69	153
Oil A	3.4	400	610	700	591	-18.7	153
Water	3.4	0.1	540	691	84.9	-0.0259	5.89
Water	3.4	100	540	691	952	-2.31	5.89
Water	3.4	200	540	691	1346	-3.27	5.89
Water	3.4	400	540	691	673	-13.1	5.89
Oil A	2	0.1	550	575	810	-0.013	583
Oil A	2	200	550	575	820	-25.7	583
Oil A	2	800	550	575	810	-104	583
Water	2	0.1	540	610	1000	-0.006	5.89
Water	2	200	540	610	1000	-12.2	5.89
Water	2	800	540	610	1000	-49.2	5.89
Oil A	0.5	0.1	575	550	816	0.013	302
Oil A	0.5	200	575	550	1152	18.3	302
Oil A	0.5	800	575	550	1140	74.1	302
Water	0.5	0.1	610	540	2000	0.003	2.74
Water	0.5	200	610	540	2000	6.06	2.74
Water	0.5	800	610	540	2000	24.2	2.74
Air	1.5	0.1	950	540	1000	1.81×10^{-4}	0.685
Air	1.5	200	950	540	1000	0.361	0.685
Air	1.5	800	950	540	1000	1.45	0.685
Helium	1.5	0.1	1460	752	1000	1.56×10^{-4}	0.722
Helium	1.5	200	1460	752	1000	0.311	0.722
Helium	1.5	800	1460	752	1000	0.795	0.722
Air	0.667	0.1	540	950	2000	-9.07×10^{-5}	0.698
Air	0.667	200	540	950	2000	-0.183	0.698
Air	0.667	800	540	950	2000	-0.724	0.698
Helium	0.667	0.1	752	1460	2000	-7.81×10^{-5}	0.728
Helium	0.667	200	752	1460	2000	-0.157	0.728
Helium	0.667	800	752	1460	2000	-0.627	0.728

There is a further complication, however. The pressure gradient due to frictional forces that would be present if the fluid were flowing at the wall temperature is, from equation (59)

$$G = -32\mu_w/Re\rho_w \qquad (15)$$

For gases, ρ_w can differ considerably from unity. As the flow starts,

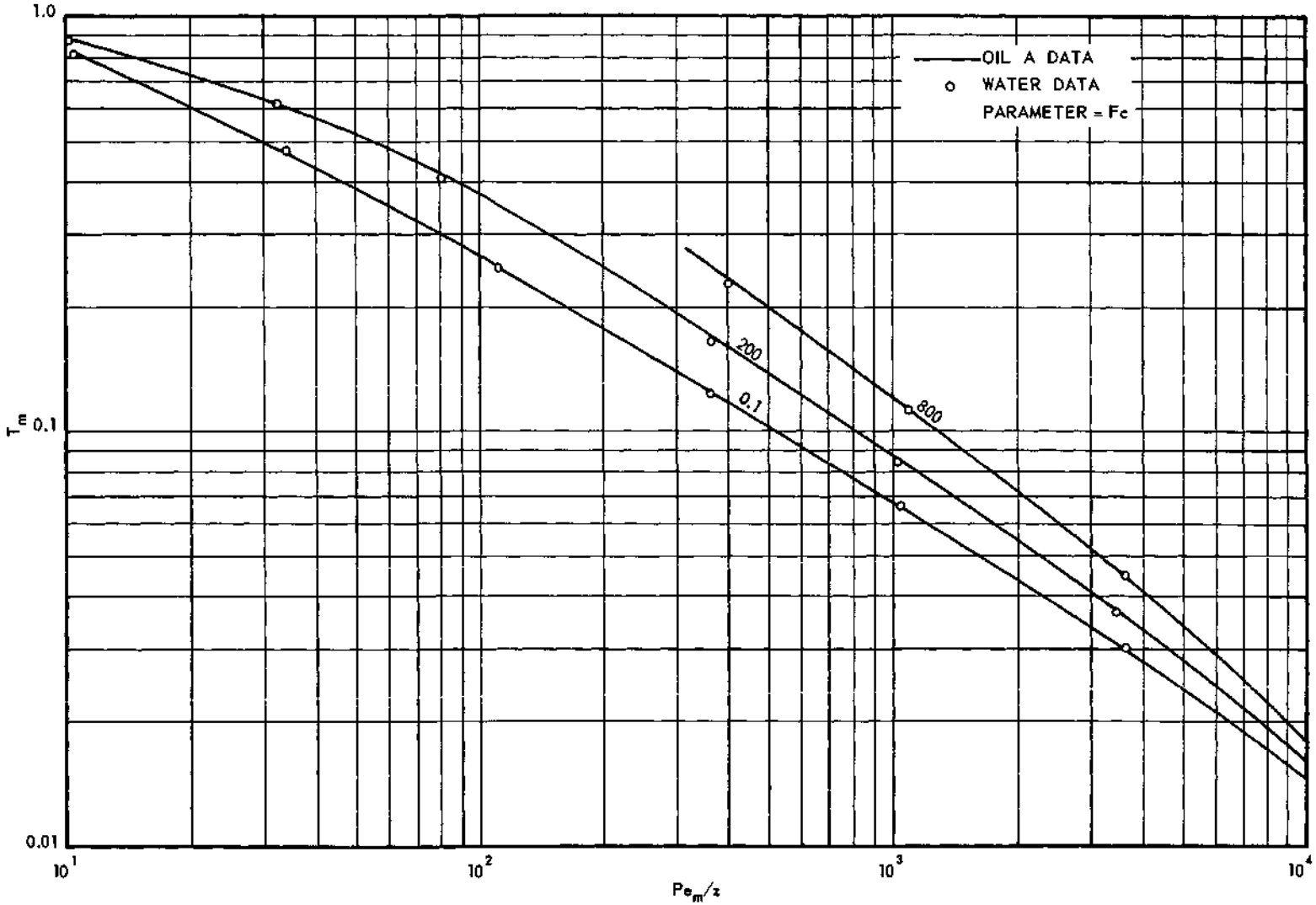

Figure 8. Comparison of Mean Temperatures for Oil A and Water, Heating in Upflow, $\mu_o/\mu_w = 2$.

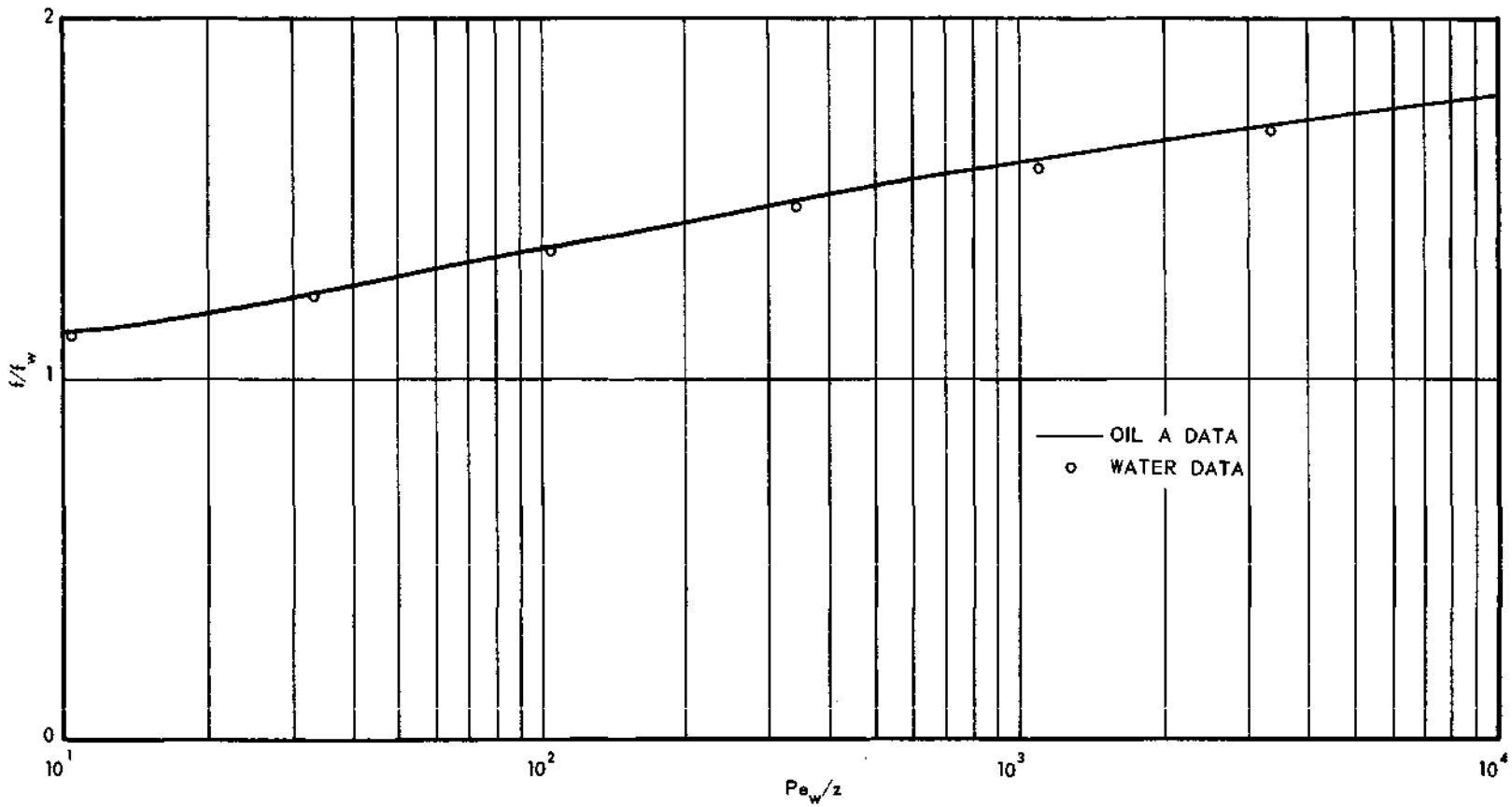

Figure 9. Comparison of Friction Factors for Oil A and Water, Heating in Upflow, $\mu_o/\mu_w = 2$.

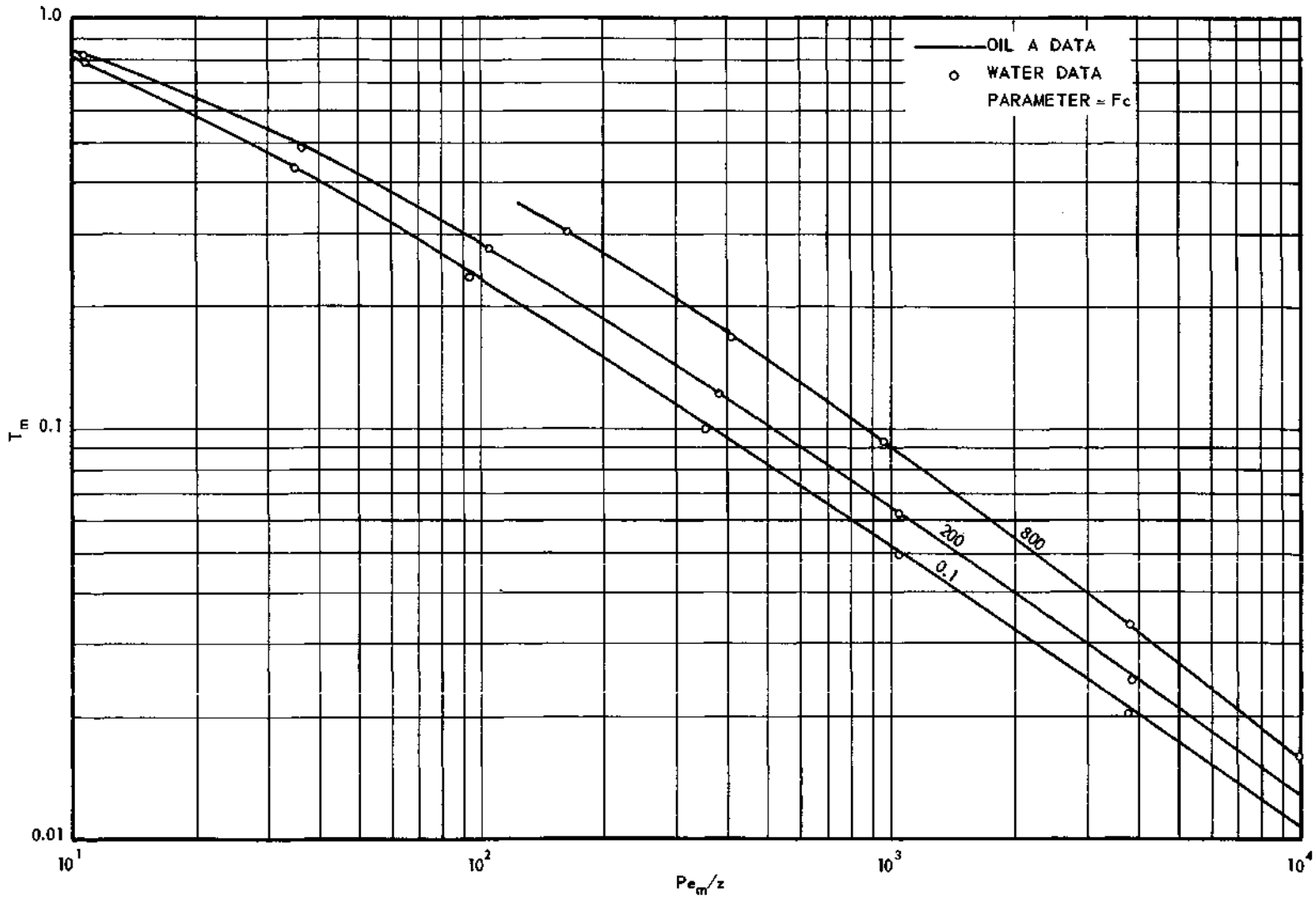

Figure 10. Comparison of Mean Temperatures for Oil A and Water, Cooling in Downflow, $\mu_o/\mu_w = 0.5$.

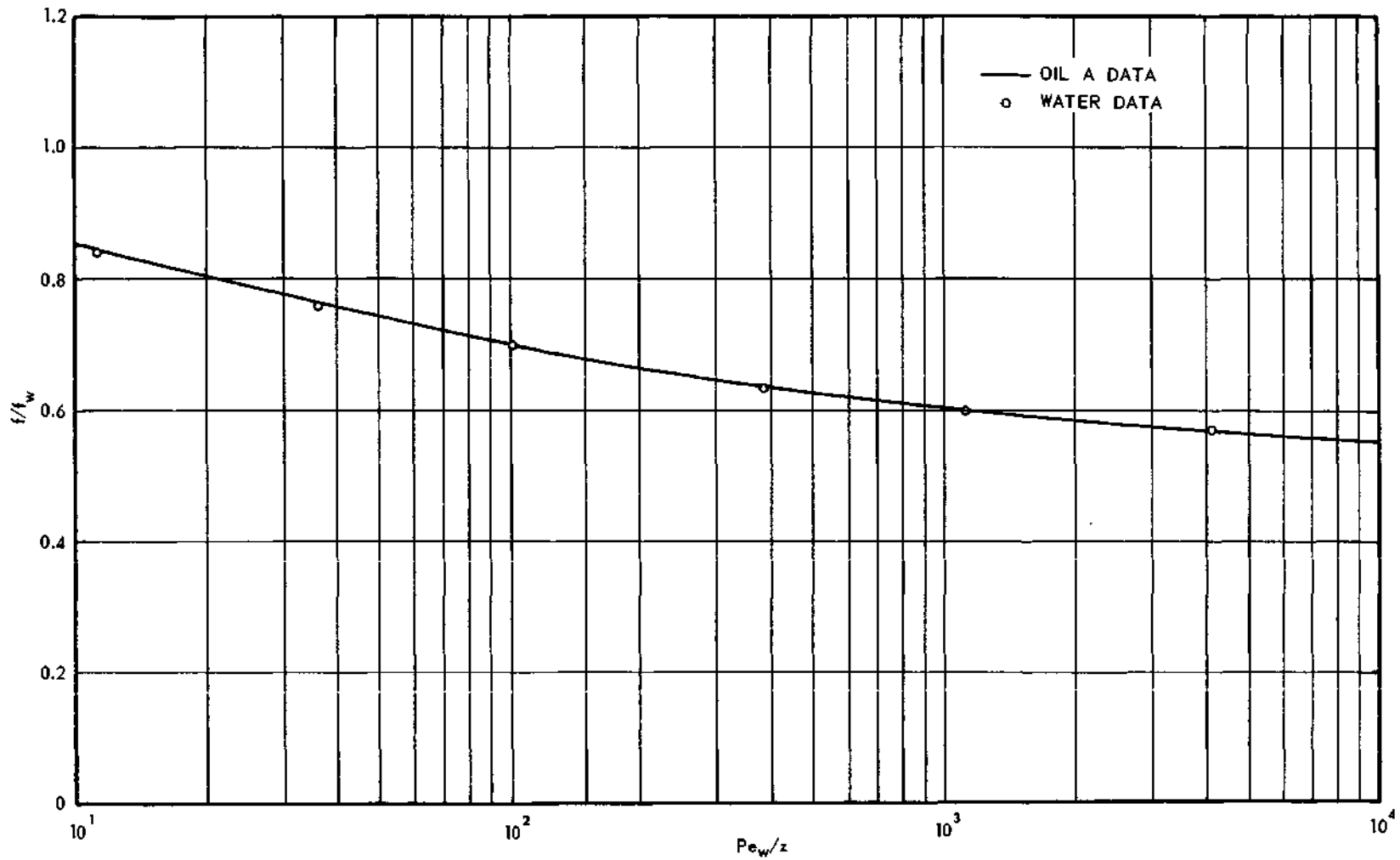

Figure 11. Comparison of Friction Factors for Oil A and Water, Cooling in Downflow, $\mu_o/\mu_w = 0.5$.

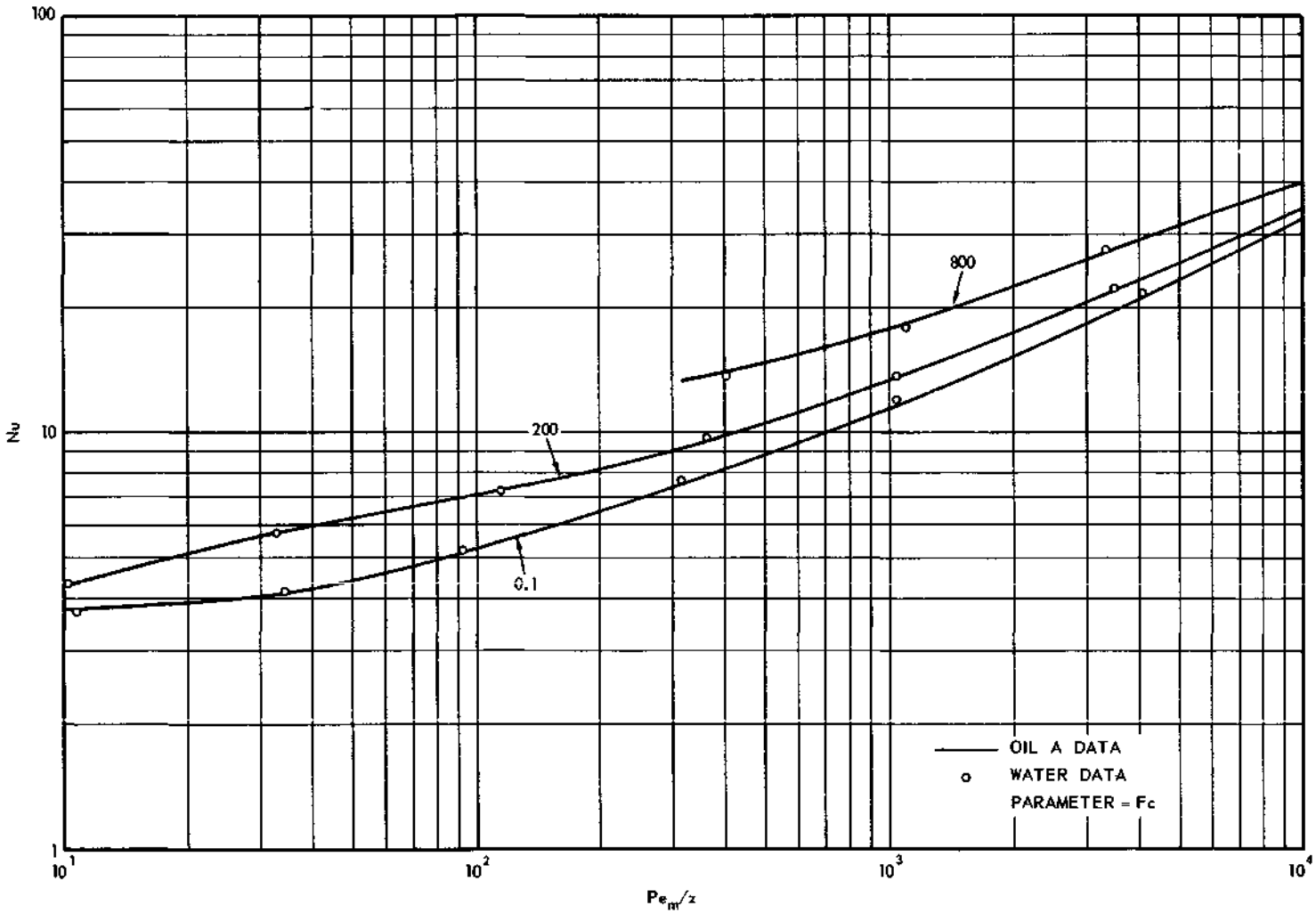

Figure 12. Comparison of Local Nusselt Numbers for Oil A and Water, Heating in Upflow, $\mu_o/\mu_w = 2$.

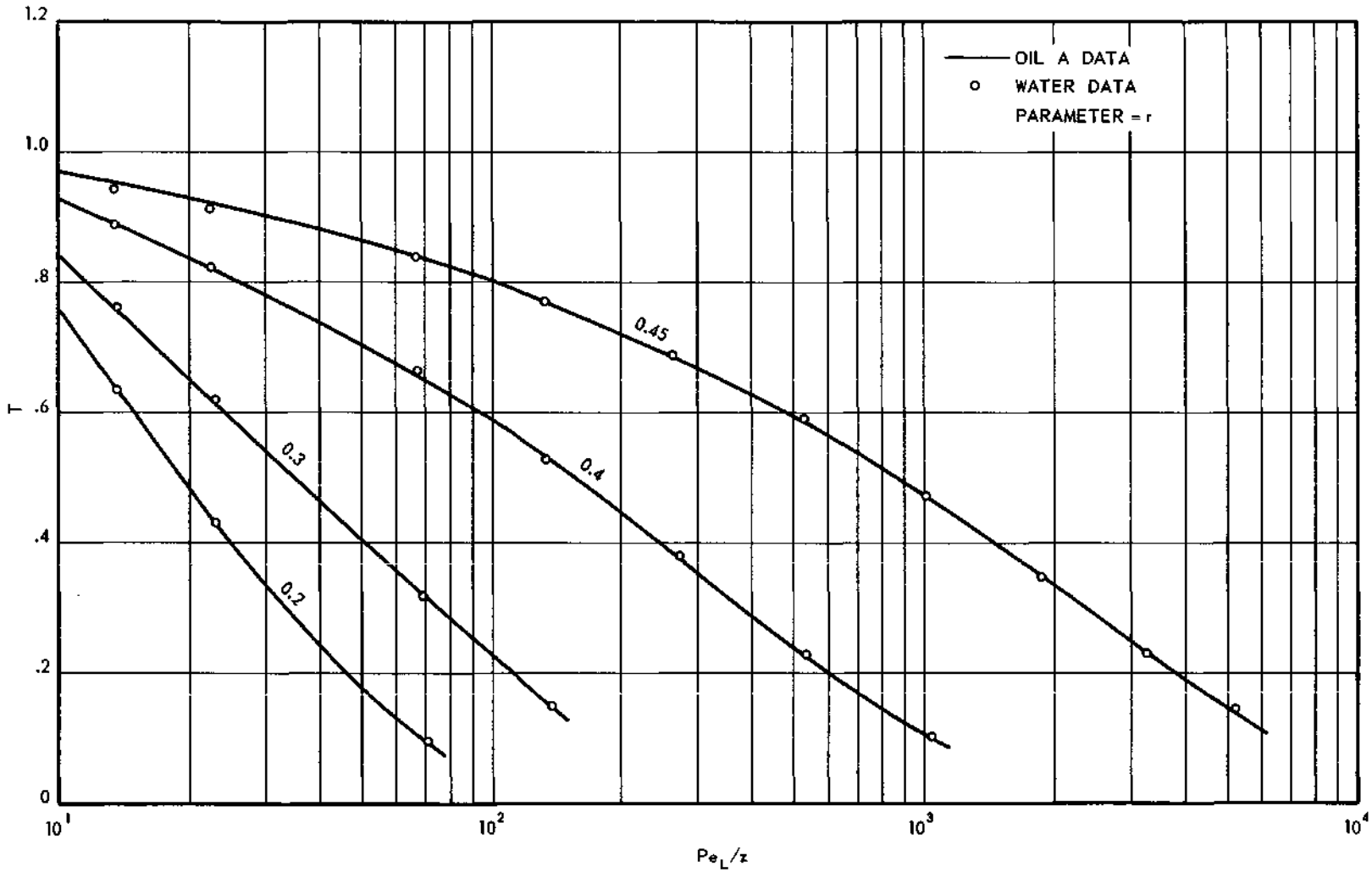

Figure 13. Comparison of Temperature Profiles for Oil A and Water, Heating in Upflow, $\mu_o/\mu_w = 2$, Fc = 0.1.

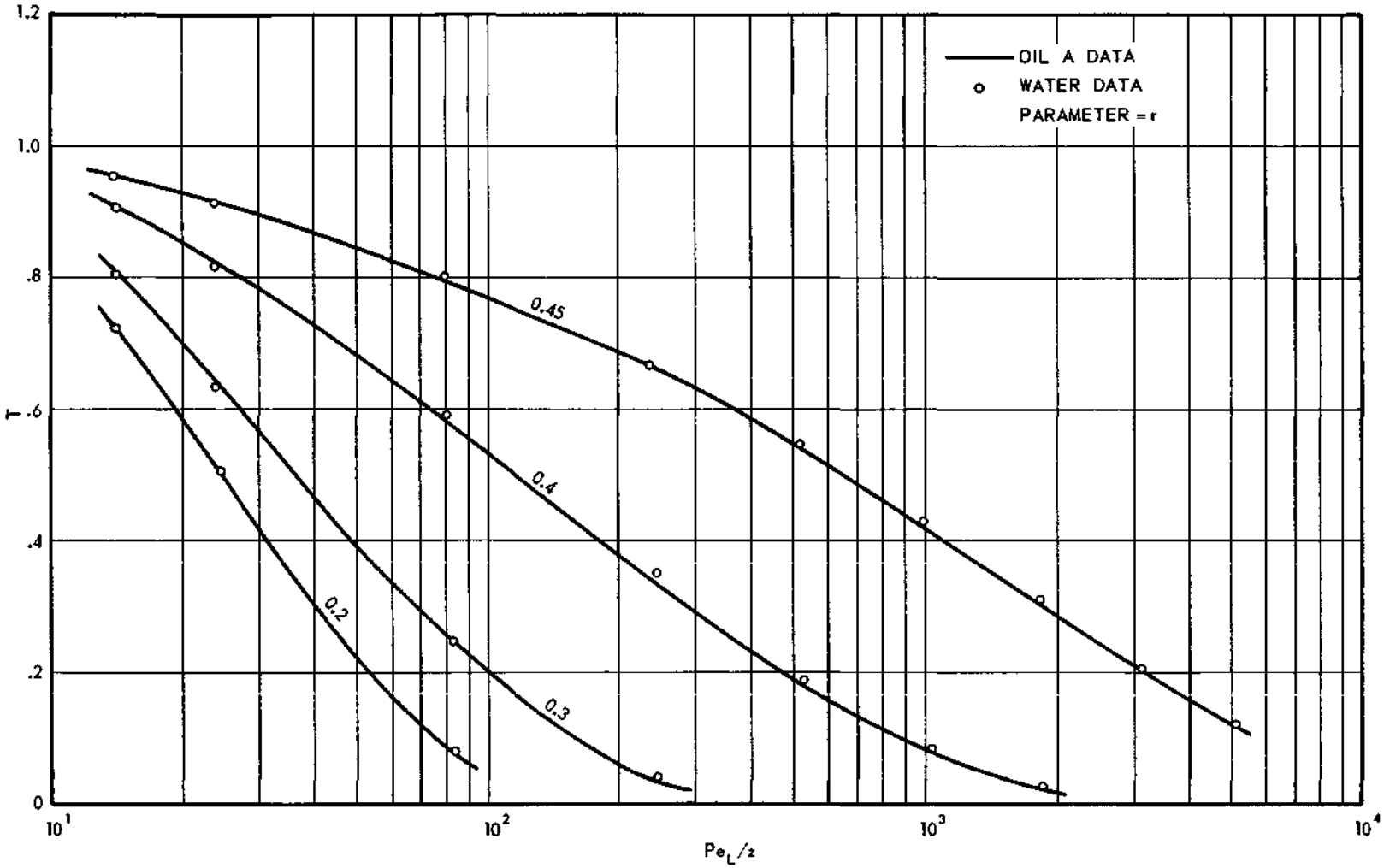

Figure 14. Comparison of Temperature Profiles for Oil A and Water, Heating in Upflow, $\mu_o/\mu_w = 2$, Fc = 200.

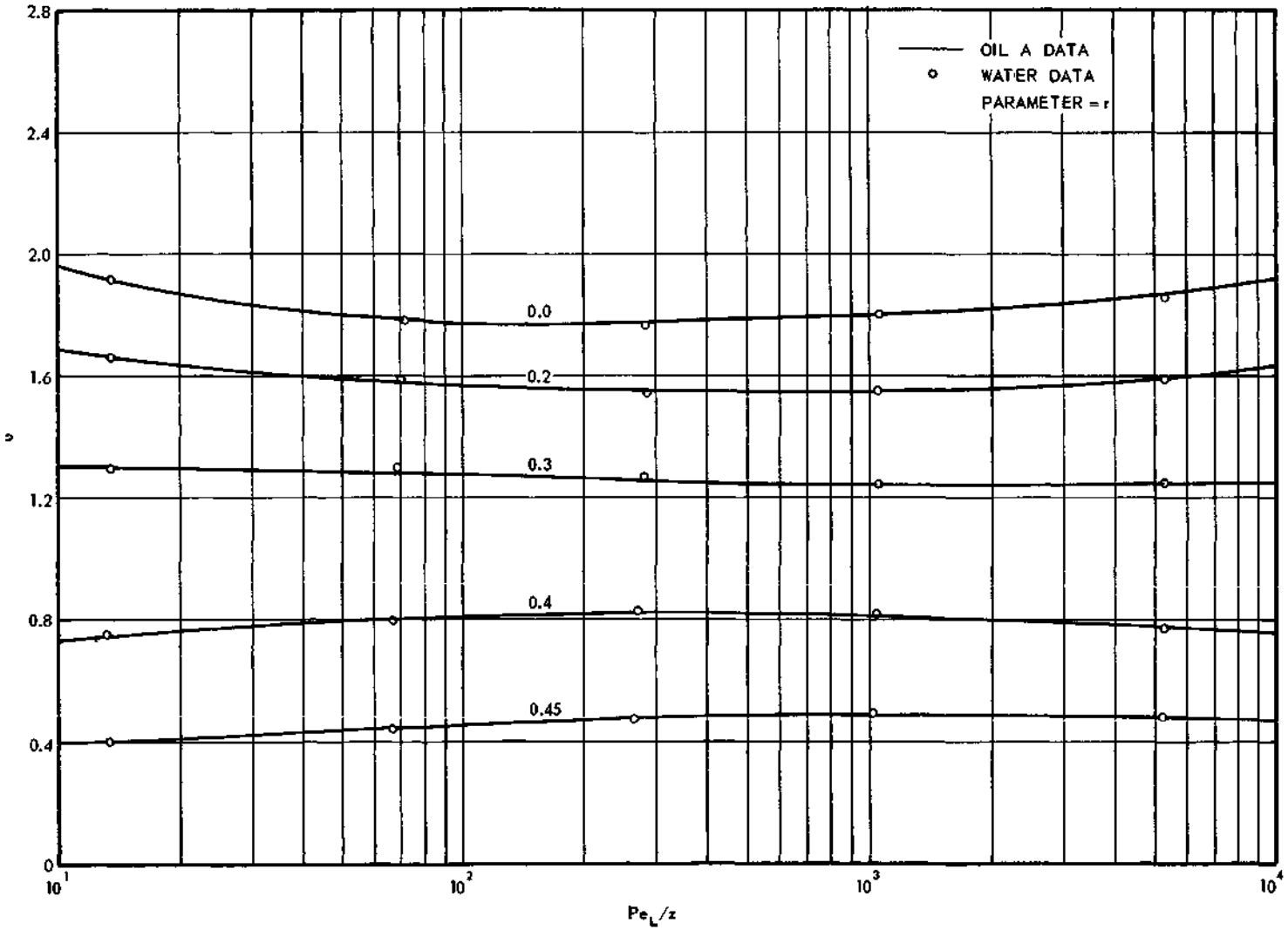

Figure 15. Comparison of Velocity Profiles for Oil A and Water, Heating in Upflow, $\mu_o/\mu_w = 2$, Fc = 0.1.

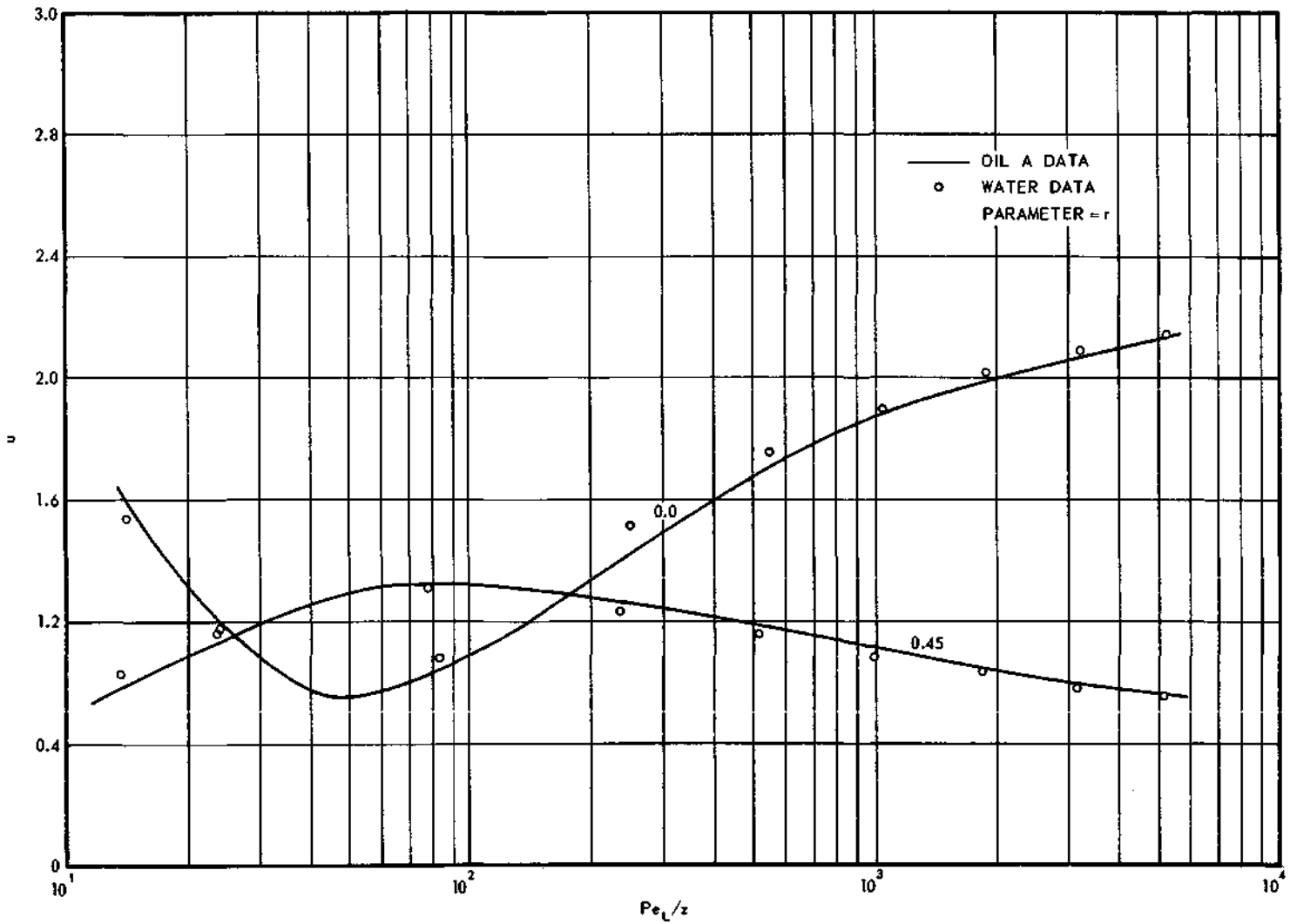

Figure 16. Comparison of Velocity Profiles for Oil A and Water, Heating in Upflow, $\mu_o/\mu_w = 2$, Fc = 200.

the gas has no idea that it will ultimately change drastically in density, and the ratio f/f_w can hardly be expected to correlate if the density ratios ρ_w differ significantly. Accordingly, it was felt that (f/f_w) x (ρ_m/ρ_w) might correlate much better for gases, which proved to be true. For liquids, the "correction ratio," ρ_m/ρ_w is of little importance; accordingly, it is not used for liquids so that the correlation may be kept as simple as possible.

In correlating velocity and temperature profiles, it is apparent that local values of Pe are required; i.e., c_p' and k' are evaluated at the temperature existing at the particular point in question. Again, there is a difference in procedure for liquids and gases. Gases, because their densities can vary so significantly, must have up correlated. For liquids, such a correction is relatively unimportant and unnecessarily complicates the correlation.

A comparison can also be made between air and oil A. It was felt that gases, because their density variation with temperature is so drastically different from that for liquids, could not be expected to give results that would correlate well with those for liquids--particularly for large temperature differences (hundreds of degrees). Because of this, and because the viscosity of gases is less sensitive than liquids to temperature, it was felt that the substances would yield comparable results only when free convection is unimportant or under the hypothetical condition $\mu_o/\mu_w = 1$. This latter condition was achieved by holding the viscosity constant in the computer program while letting the other properties vary in their normal manner. The temperature change for air was limited to 90° F, in which range the deviation of density from a

linear function was not too drastic (but still significant).

It should be noted that a liquid heated in upflow, for example, must be compared with a gas cooled in downflow because the effect of temperature on the viscosities of the two classes of substances is in opposite directions--the viscosity of a liquid decreases with increasing temperature, while the viscosity of a gas increases with increasing temperature. Accordingly, heating must be compared with cooling, and, then, to have free-convection effects either aiding or opposing forced-convection effects in both cases, upflow must be compared with downflow.

Comparison of the data in the tables in Appendix F (discussed in the next section) reveals that, as expected, results for liquids and gases do coincide when free convection is unimportant ($Fc = \pm 0.1$) and when $\mu_o/\mu_w = 1$; in other cases, the data correlated poorly. As an example of this comparison, mean temperatures are compared in Figure 17.

Because liquids and gases cannot be expected to yield comparable results when the temperature range for gases is several hundred degrees, it is necessary to consider gases separately. It was attempted to show that, at least, the results for different gases can be correlated with each other.

Air and helium, which are relatively dissimilar gases (see Appendix E) were compared under the conditions shown in Table 4. Results comparable to those for liquids were obtained; for example, Figures 18-22 show typical comparisons of pertinent quantities for gases.

Radial Velocity and Initial Profile Shape.--It will be recalled that in

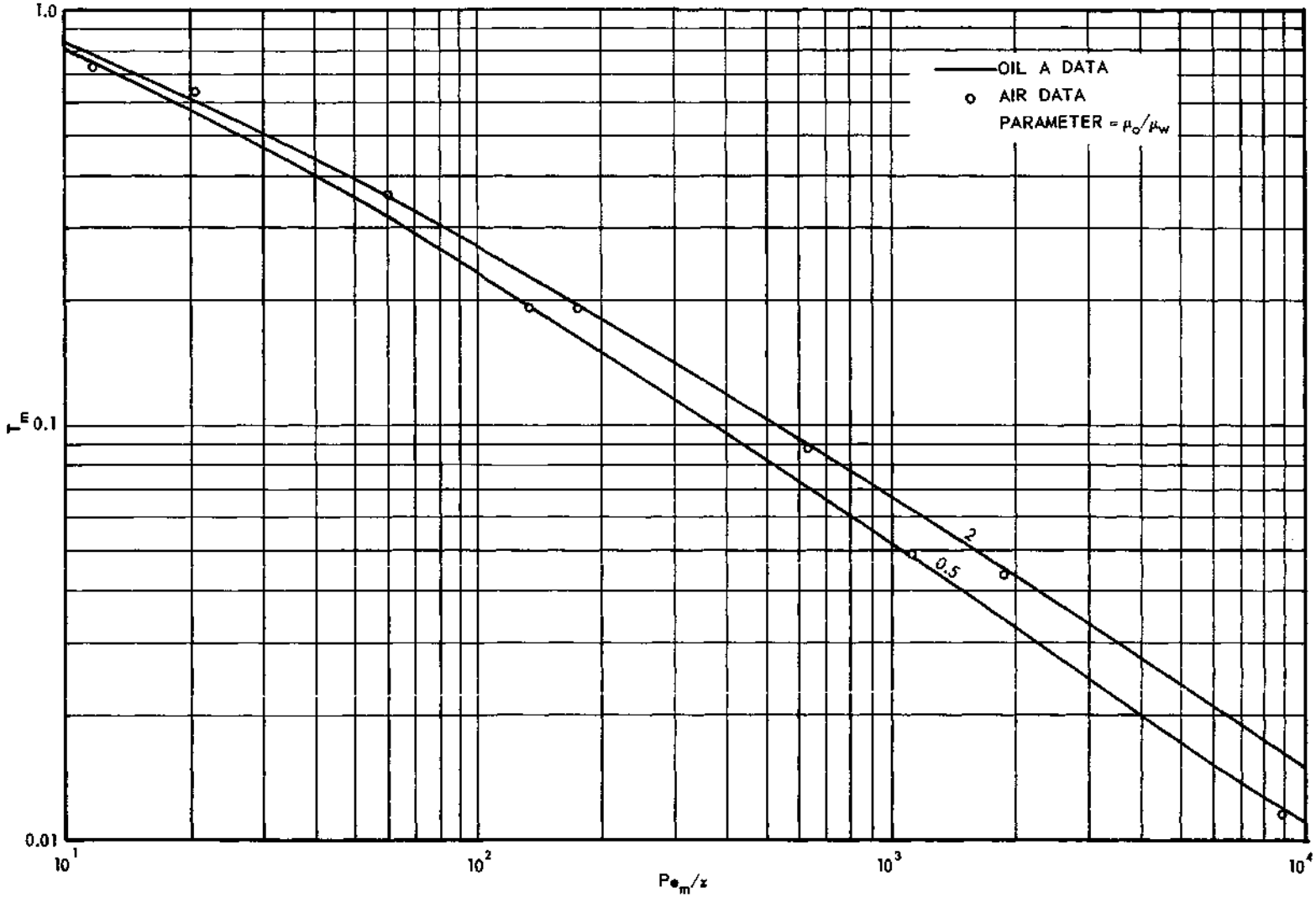

Figure 17. Comparison of Mean Temperatures for Oil A and Air, Fc = 0.1.

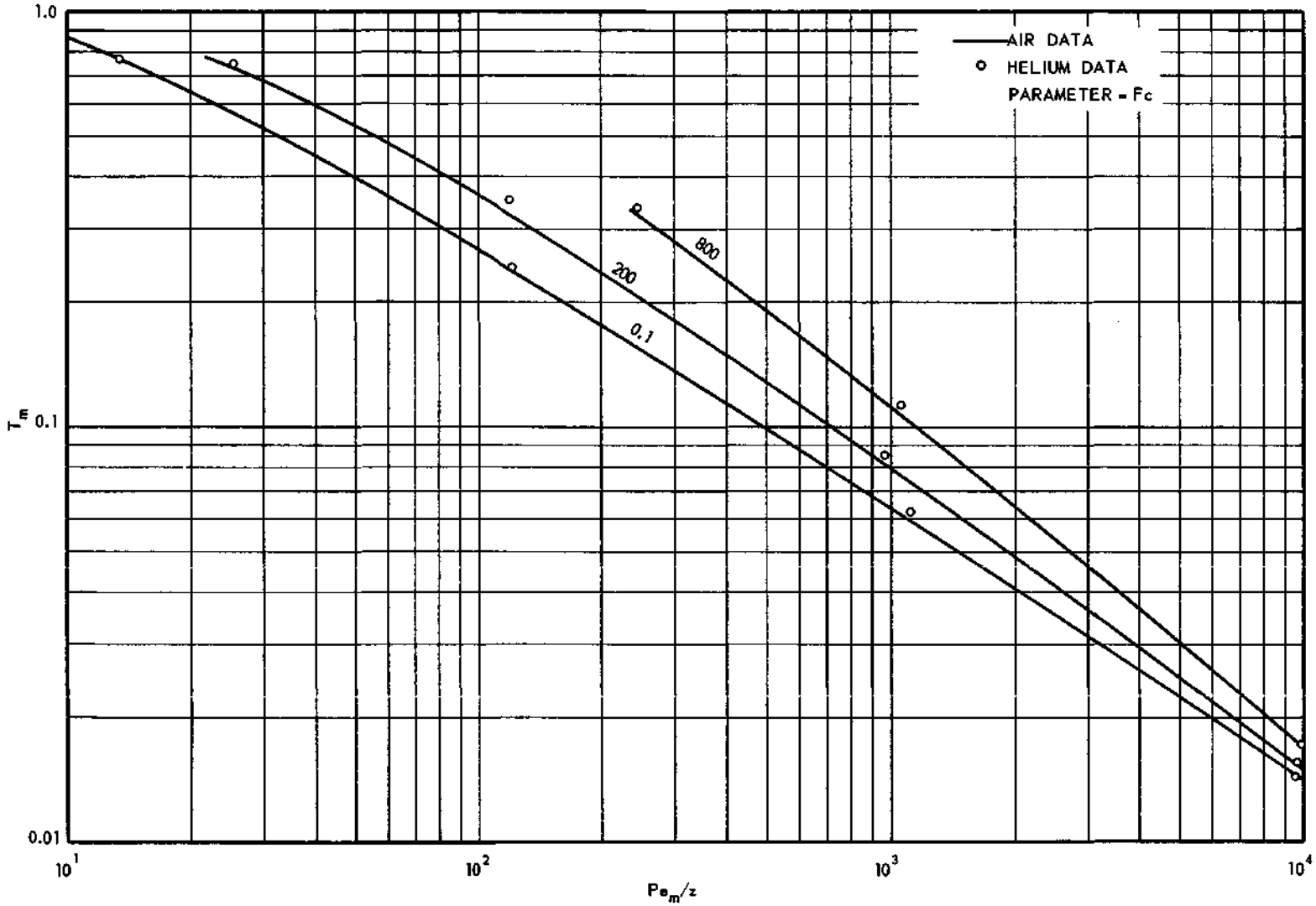

Figure 18. Comparison of Mean Temperatures for Air and Helium, Cooling in Downflow, $\mu_o/\mu_w = 1.5$.

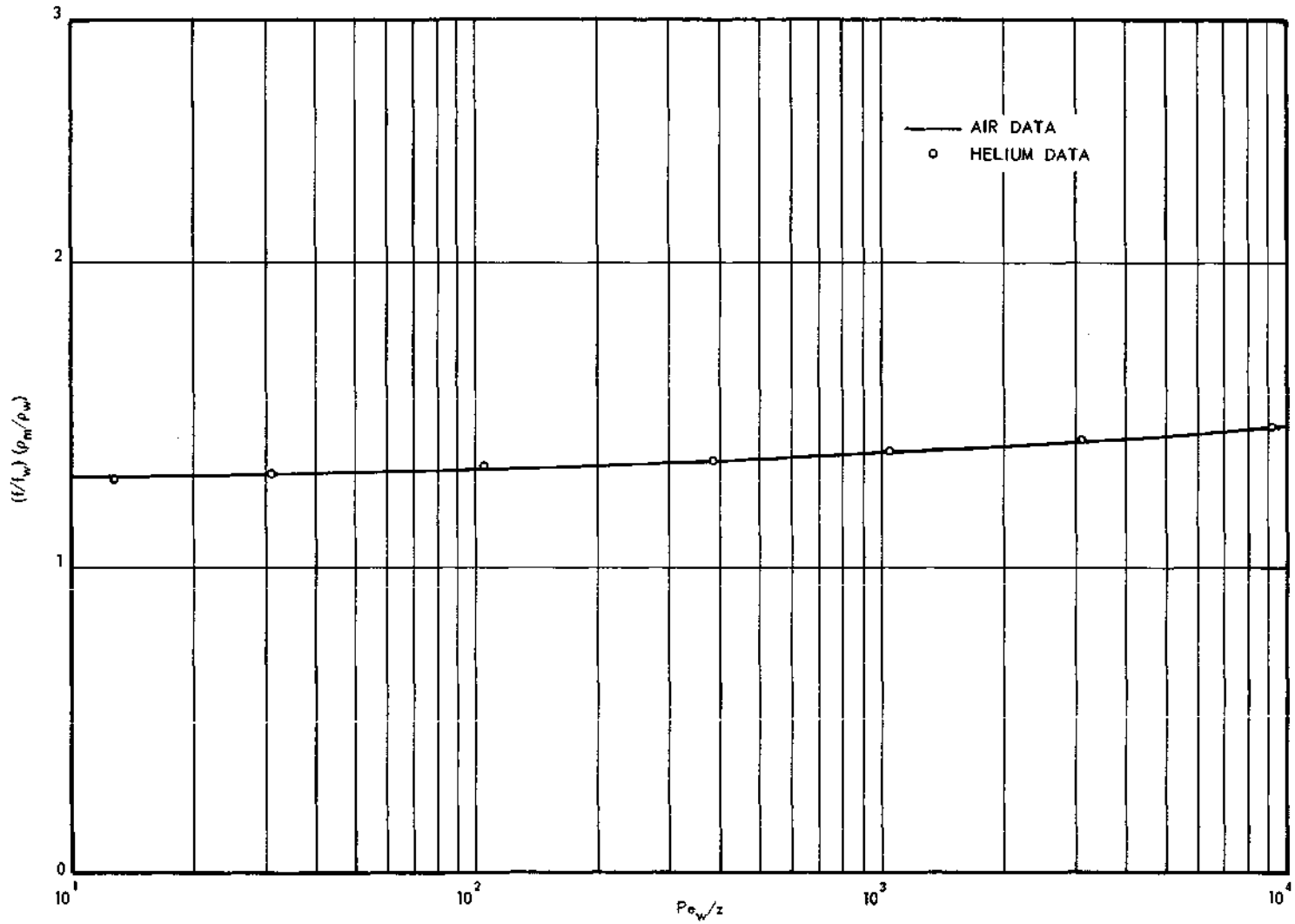

Figure 19. Comparison of Friction Factors for Air and Helium, Cooling in Downflow, $\mu_o/\mu_w = 1.5$.

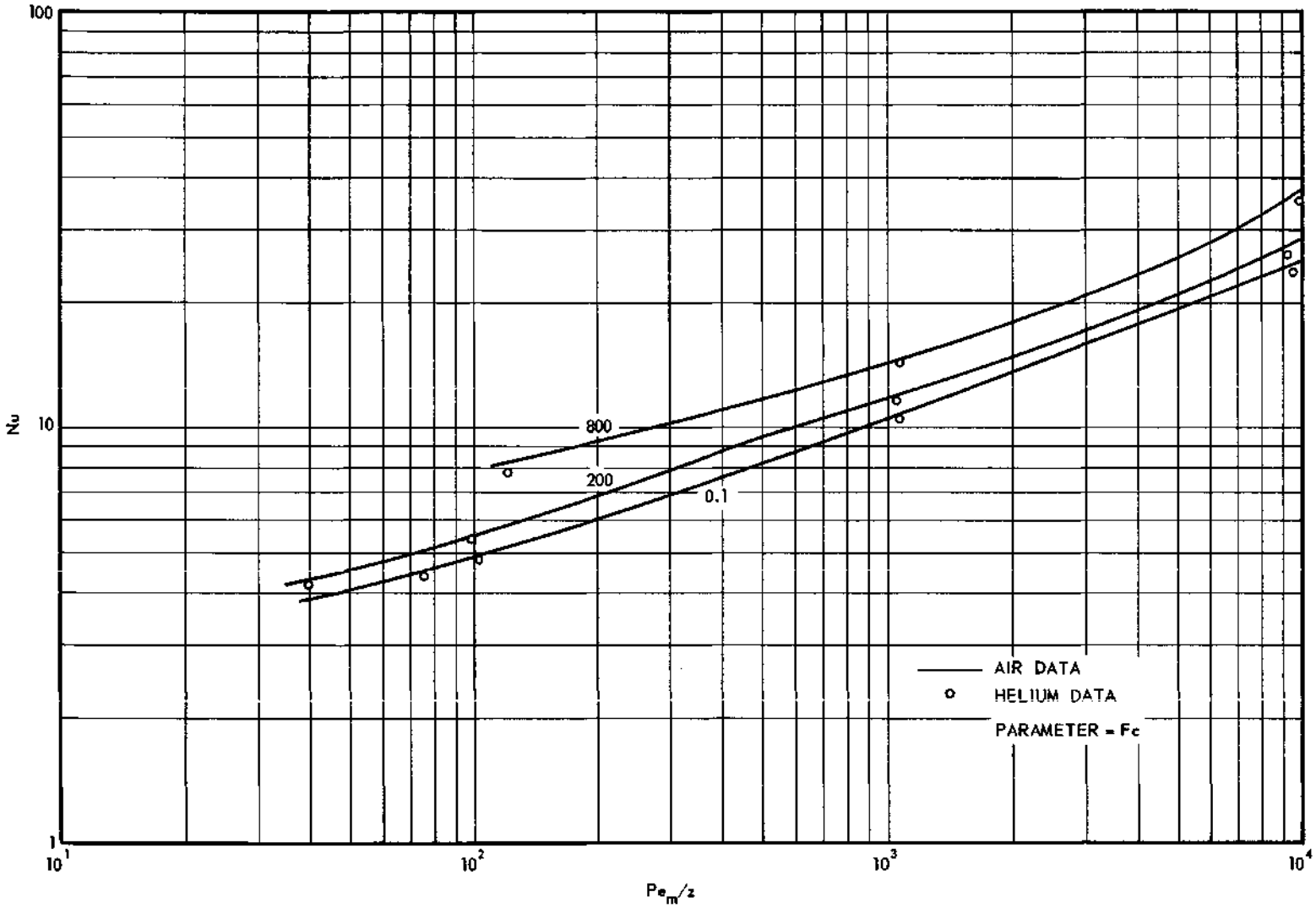

Figure 20. Comparison of Local Nusselt Numbers for Air and Helium, Heating in Upflow, $\mu_o/\mu_w = 0.667$.

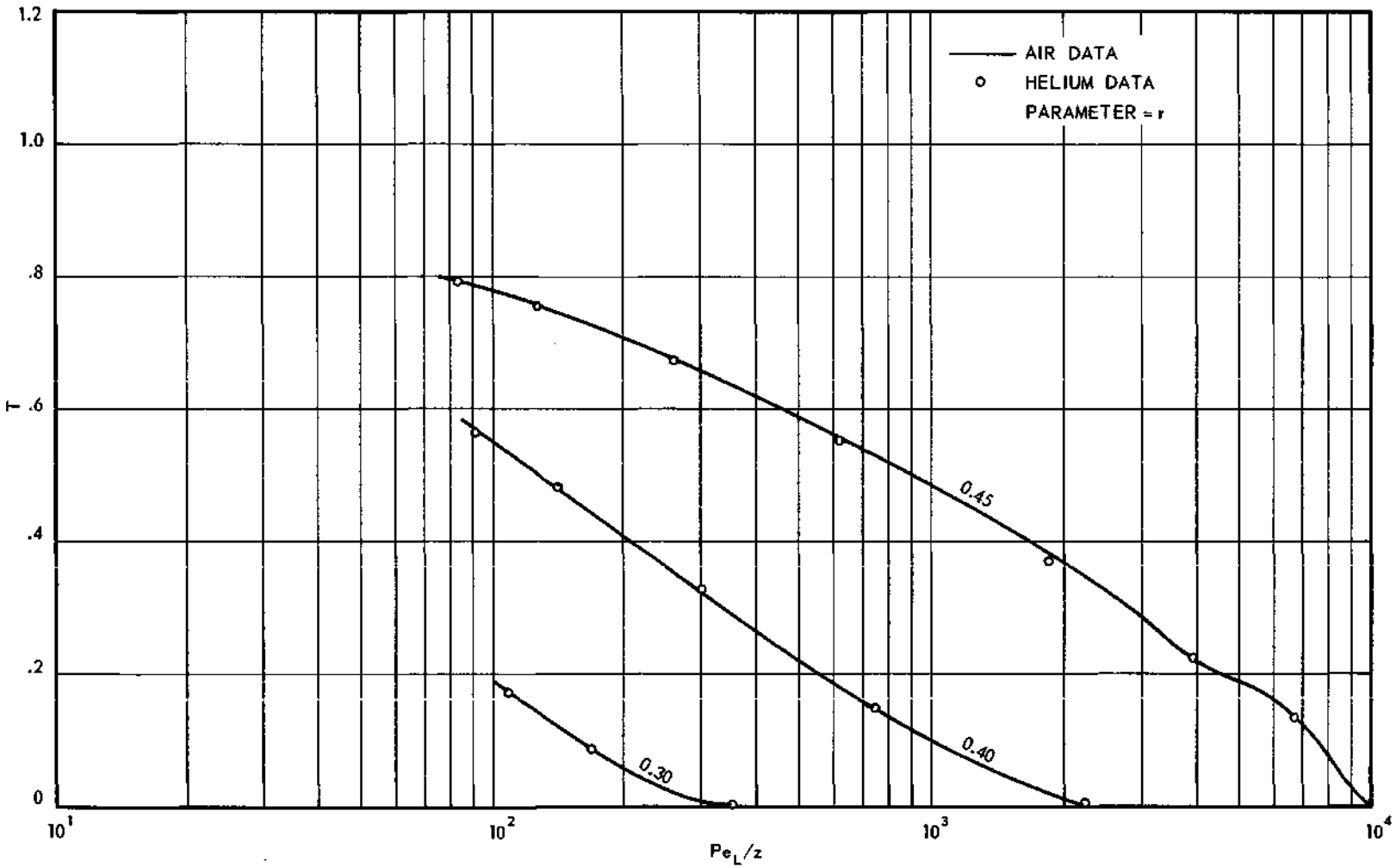

Figure 21. Comparison of Temperature Profiles for Air and Helium, $\mu_o/\mu_w = 0.667$, Fc = 800.

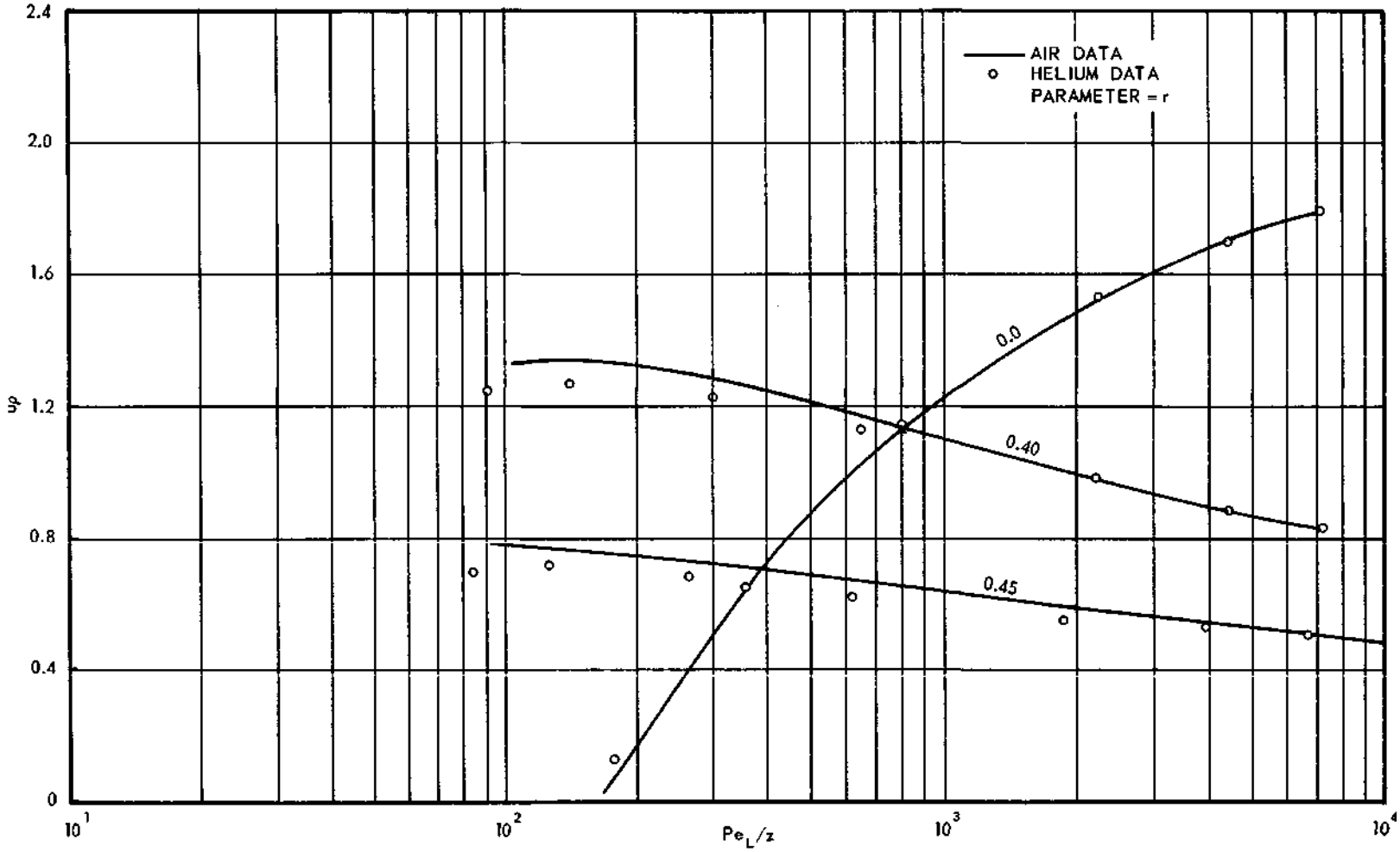

Figure 22. Comparison of Velocity Profiles for Air and Helium, Heating in Upflow, $\mu_o/\mu_w = 0.667$, Fc = 800.

Table 5. Typical Magnitudes of Radial Velocity Components.

Substance	μ_o/μ_w	Fc	Maximum v at $z = (\Delta z)_o$	Maximum v at $z = 8(\Delta z)_o$
Oil A	1	100	4.3×10^{-3}	3.4×10^{-5}
Oil A	1	200	1.7×10^{-2}	1.7×10^{-4}
Oil A	1	800	3.4×10^{-2}	2.3×10^{-4}
Oil A	2	800	1.0×10^{-1}	3.4×10^{-4}
Oil A	5	800	2.7×10^{-1}	3.4×10^{-4}
Oil A	10	800	4.3×10^{-1}	2.6×10^{-4}
Oil A	20	0.1	3.4×10^{-1}	6.3×10^{-5}
Oil A	20	200	4.0×10^{-1}	8.3×10^{-5}
Oil A	20	800	6.3×10^{-1}	2.1×10^{-4}
Oil A	0.05	800	-4.0×10^{-2}	-3.4×10^{-3}
Oil A	20	-0.1	3.4×10^{-1}	6.3×10^{-5}
Oil A	0.05	-0.1	-4.6×10^{-2}	-1.2×10^{-2}
Air	1	800	3.0	1.31×10^{-1}
Air	0.667	0.1	-1.4	-6.0×10^{-2}
Air	0.5	0.1	-2.2	-1.8×10^{-1}

this analysis radial velocity terms were assumed to be quite small. As a check on the validity of this assumption, the scheme described by equation (109) was used to estimate the magnitude of the radial velocity components. The results indicate that at $z = (\Delta z)_o$, the radial velocity terms can be appreciable (of the order of v = 0.6 for oil A; of the order of v = 3 for air--at worst). Typical results are given in Table 5. It must be realized that these values of v are only qualitative.

In addition, it is quite possible their magnitudes are due to inadequacy of the numerical scheme near $z = (\Delta z)_o$. After a few steps down the tube (at which Pe/z is still quite large--10^5 or more), the radial velocity components rapidly tend to become quite negligible (of the order of 10^{-4} to 10^{-5}) and their derivatives appear likewise to be negligible.

The appearance of the large radial velocity terms near the tube entrance is not considered to be disastrous. These components arise because a parabolic profile is suddenly changed in shape. Radial velocity components would tend to make this change more gradual than observed in this study, perhaps, but because of large viscosity and/or free-convection forces, it appears that this adjustment must take place quite rapidly in any event. In the meantime, the temperature profile is almost completely undeveloped, and, therefore, is hardly affected. This fact is important, and for emphasis it will be stated in a different way: during the initial few steps, the temperature profile develops hardly at all, and it is therefore virtually independent of what happens to the velocity profile because of viscosity variations or free-convection forces. Justification for this statement is given in Table 6, in which temperatures at $r = 0.45$ are given for values of z near the heating section entrance.

The calculations described in Table 6 are for oil A, with heating in upflow. They are also typical of results for other flow conditions. Also given are the maximum value of v occurring at the value of z in question and the value of u at $r = 0$, which is indicative of the extent to which the velocity profile has flattened. It is evident that at this value of z, the temperature T is still virtually zero at $r = 0.45$ (and

Table 6. Radial Velocity Components near Heating Section Entrance for Calculations with Oil A.

z	Pe/z	T at r = 0.45	μ_o/μ_w	Fc	Maximum v	Center-line Velocity	$(\Delta z)_o$
1.6	2.95×10^5	9.78×10^{-5}	2	100	1.8×10^{-5}	1.92	0.1
1.6	4.49×10^5	2.0×10^{-5}	5	200	1.5×10^{-5}	1.74	0.1
1.6	2.66×10^5	2.80×10^{-5}	10	400	3.0×10^{-4}	1.46	0.1
1.6	3.80×10^5	5.70×10^{-6}	20	800	2.1×10^{-4}	0.987	0.1

consequently at all values of r except very near the tube wall) and that the radial velocity components have become quite small. Yet, it is apparent that considerable flattening in the velocity profile has already taken place. This is possible because the Prandtl number for oil A is so large. Similar results can be expected for any liquid with large Prandtl number, since the larger the Prandtl number, the faster the velocity profile develops compared to the temperature profile.

This result has important implications: for oils, the shape of the initial velocity profile is apparently immaterial--the flow soon adjusts to the conditions imposed by viscosity and free-convection forces, and changes very slowly thereafter. Thus, the results presented in this study for liquids may be expected to hold for cases of arbitrary initial velocity profile--which in many cases will be neither flat nor parabolic.

For gases, this conclusion does not hold. In no case was the profile distorted appreciably from a parabolic one in the initial few steps; accordingly, there is no evidence that the results are independent of

initial profile shape. (If it were attempted to go from a flat profile to a near parabolic one in a short distance, enormous radial velocity terms would result, which could certainly not be neglected.) In fact, there is considerable evidence that the initial profile shape is very important. Accordingly, the results of this study must be applied to gases only in cases in which the profile is fully developed at the entrance to the heating section.

A fluid such as water, with a Prandtl number in the range of 5 to 10 (for some temperatures) is an intermediate case. On the basis of the few calculations with water in this study, it appears that in the length it takes the velocity profile to change significantly and for radial velocity terms to become negligible, the temperature profile has developed to some extent, but not a great deal. (In a typical case, the temperature T ar r = 0.45 was 0.110 at a point where the radial velocity had just become negligible.) It will be assumed satisfactory to apply the results of this study to arbitrary initial velocity profiles for any fluid with $Pr \geq 5$, but admittedly, this is somewhat open to question.

Correlated Results.--Correlated results are given in Appendix F. These results are intended to be used in those cases where direct use of the numerical scheme developed above is impossible, impractical, or undesirable.

Two sets of results are given: those for liquids, and those for gases. All liquid results were obtained using the physical properties of oil A, since it has been shown that the results, when considered to be functions of the appropriate variables, may be reasonably expected

to apply to common liquids. Likewise, the results for gases were obtained using the physical properties of air.

An attempt was made to have the coverage complete enough to cover most cases of interest. Heating in upflow, cooling in downflow, heating in downflow, and cooling in upflow were considered. The latter two cases are probably of little practical value, because free-convection effects oppose the forced-convection effects. Nevertheless, these cases were included because there may be instances where other considerations dictate that these types of flows be used.

Free-convection parameters of ± 0.1 (free convection negligible), 100, 200, 400, 800, -100, and -200 were considered. Actually, free convection parameters of -400 and -800 were likewise investigated, but it was found that in all cases the flow became unstable at $Pe/z \geq 3 \times 10^4$, so these results are not reported. An upper limit of 800 for Fc seems to be a practical one, because above this value (and even somewhat below it), flows tend to become unstable at rather large values of Pe/z--and in this region, Pigford's analysis is probably satisfactory. Viscosity ratios μ_o/μ_w of 0.05, 0.1, 0.2, 0.5, 1, 2, 5, 10, and 20 were considered for liquids, and viscosity ratios of 0.5, 0.667, 1, 1.5, and 2 were considered for gases. Results are reported for $10 \leq Pe/z \leq 10^4$. For smaller values of Pe/z, asymptotic behavior (fluid approaching wall temperature) is rapidly approached, and larger values do not appear to be of significant interest (although, if they are, Pigford's analysis should be satisfactory in this region).

A convenient method for computing T_m is based on the data in Figures 23 through 28. Figure 23 shows the effect of viscosity alone

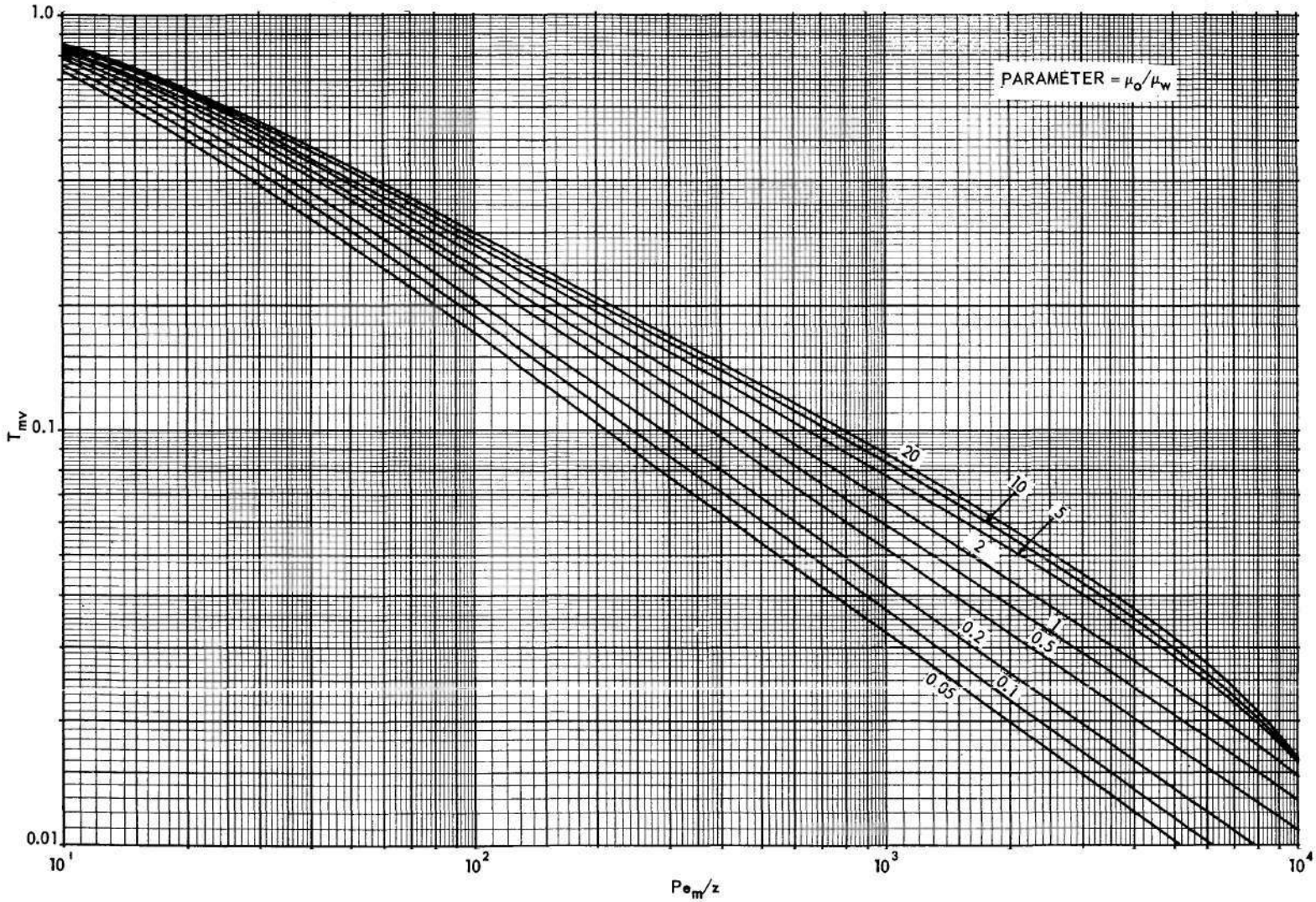

Figure 23. Effect of Viscosity on Mean Temperature.

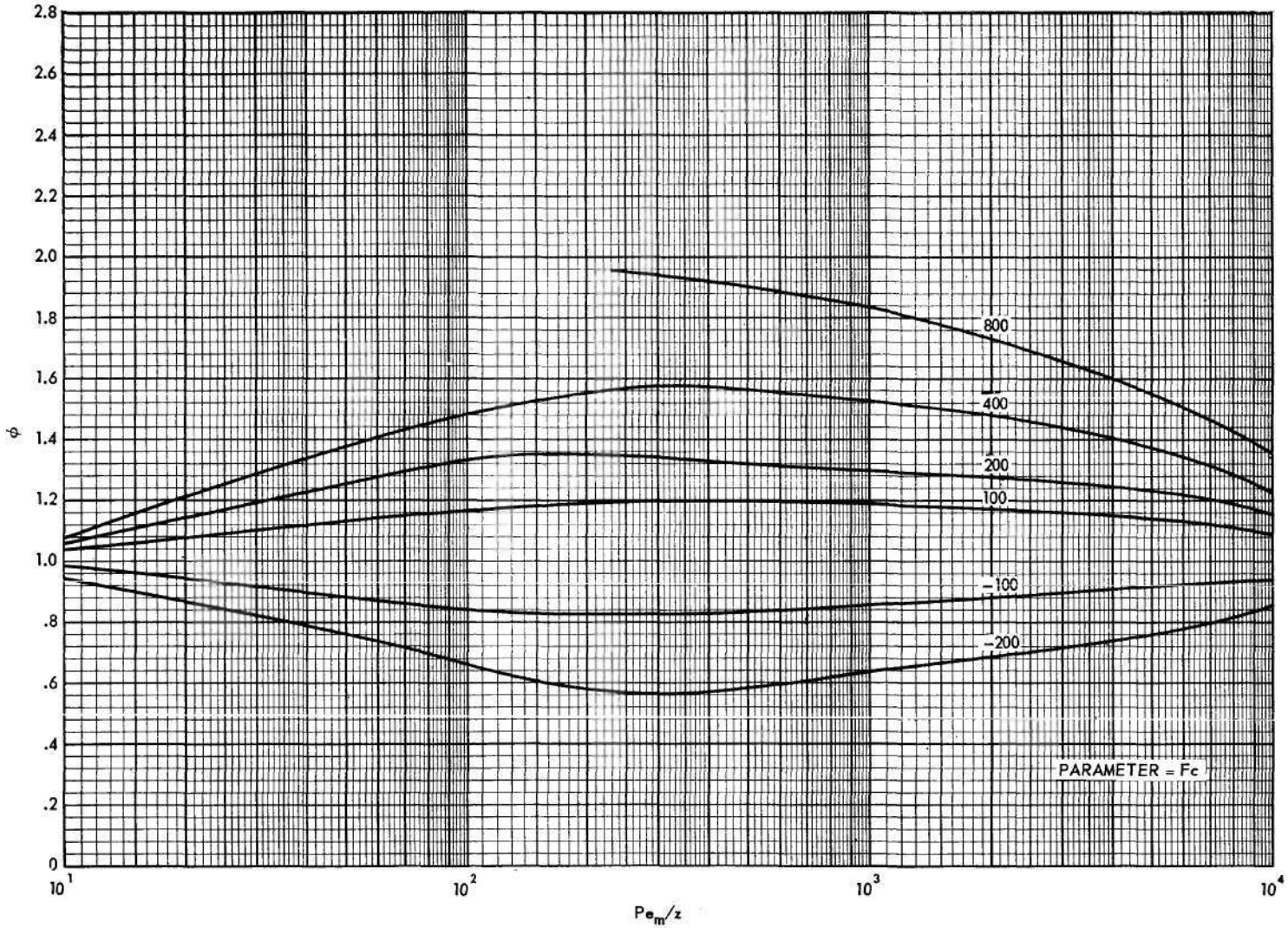

Figure 24. Correction Factor for Effect of Free Convection on Mean Temperature.

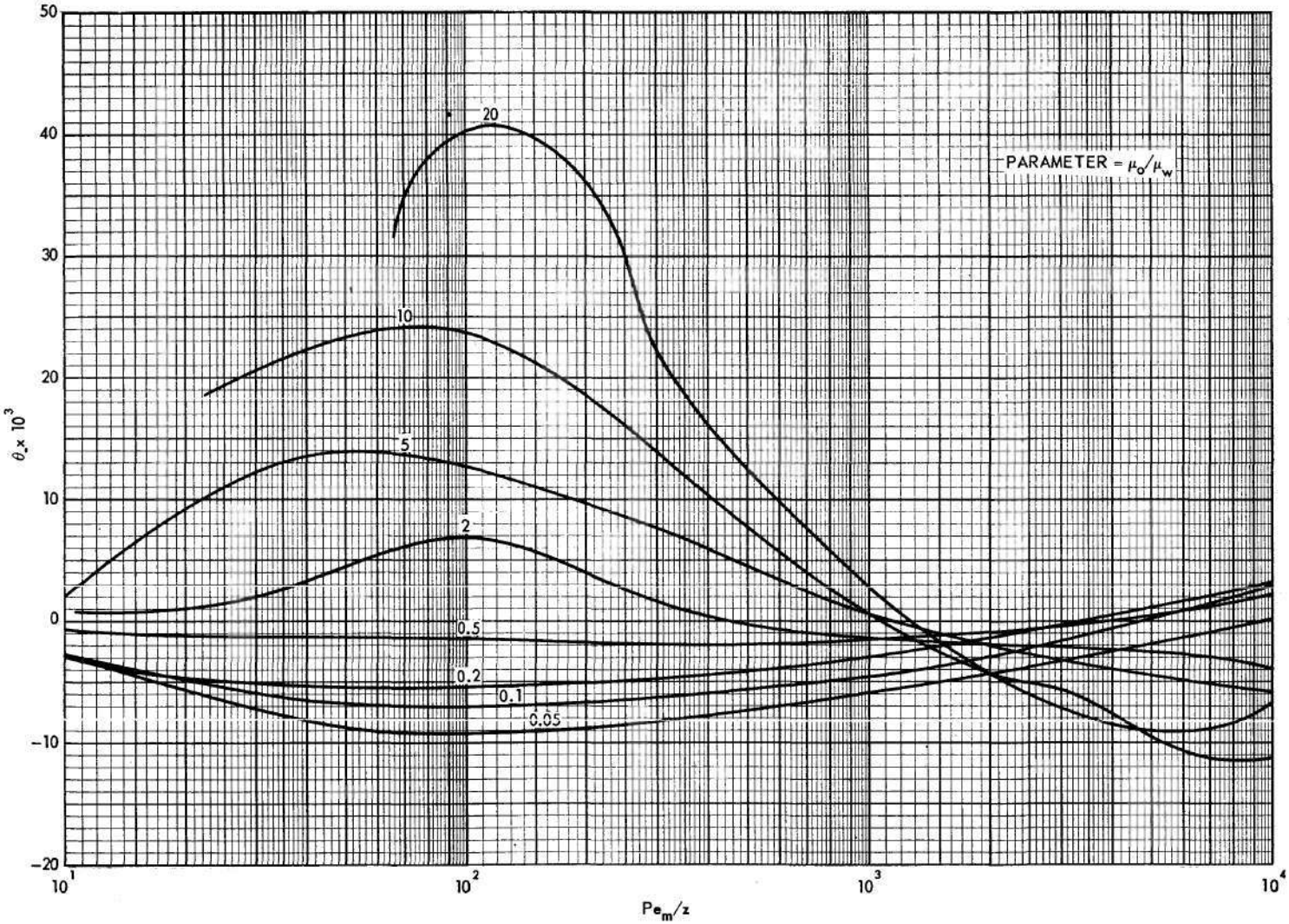

Figure 25. Correction Factor for Effect of Viscosity on Free Convection for Liquids with Positive Fc.

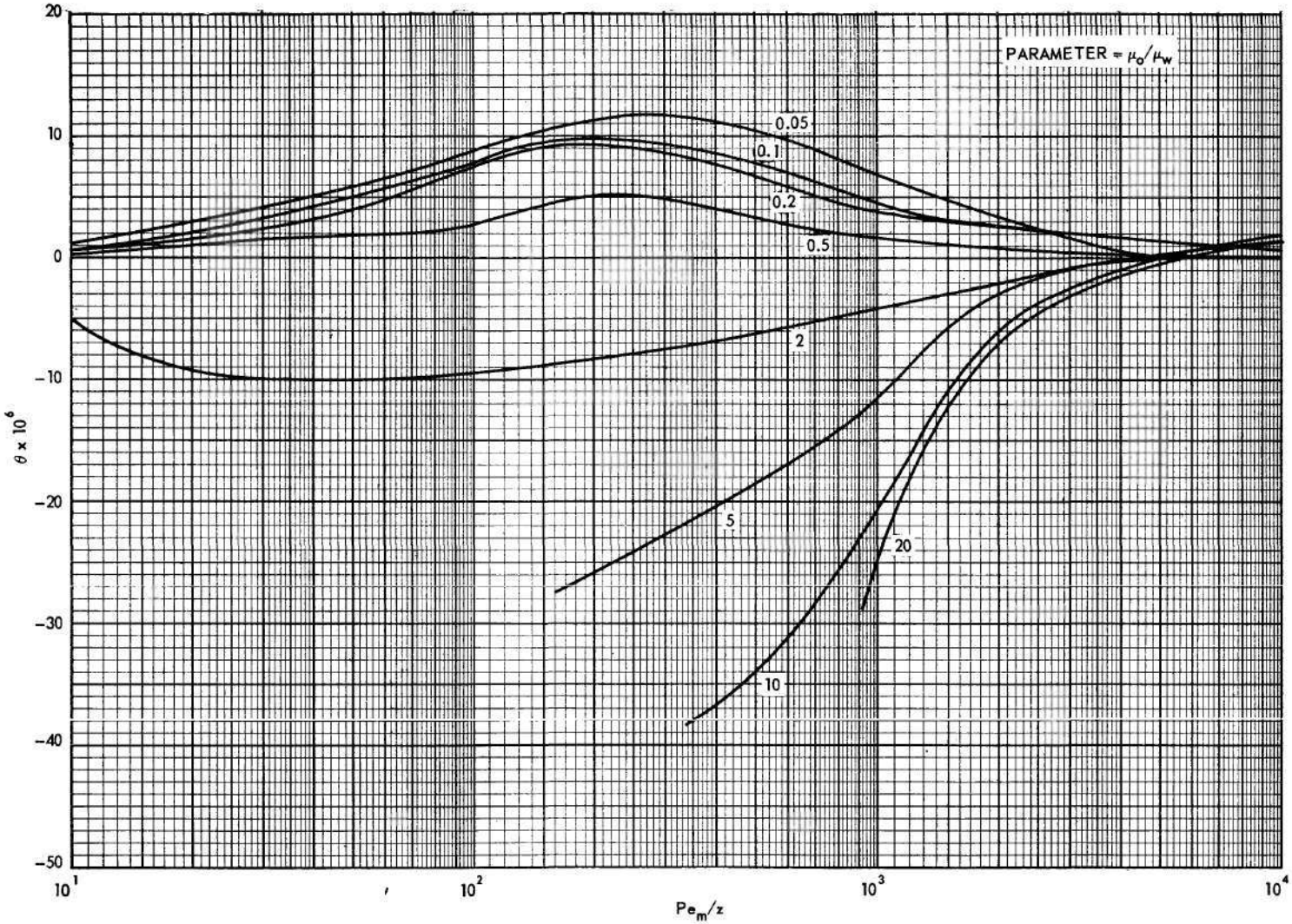

Figure 26. Correction Factor for Effect of Viscosity on Free Convection for Liquids with Negative Fc.

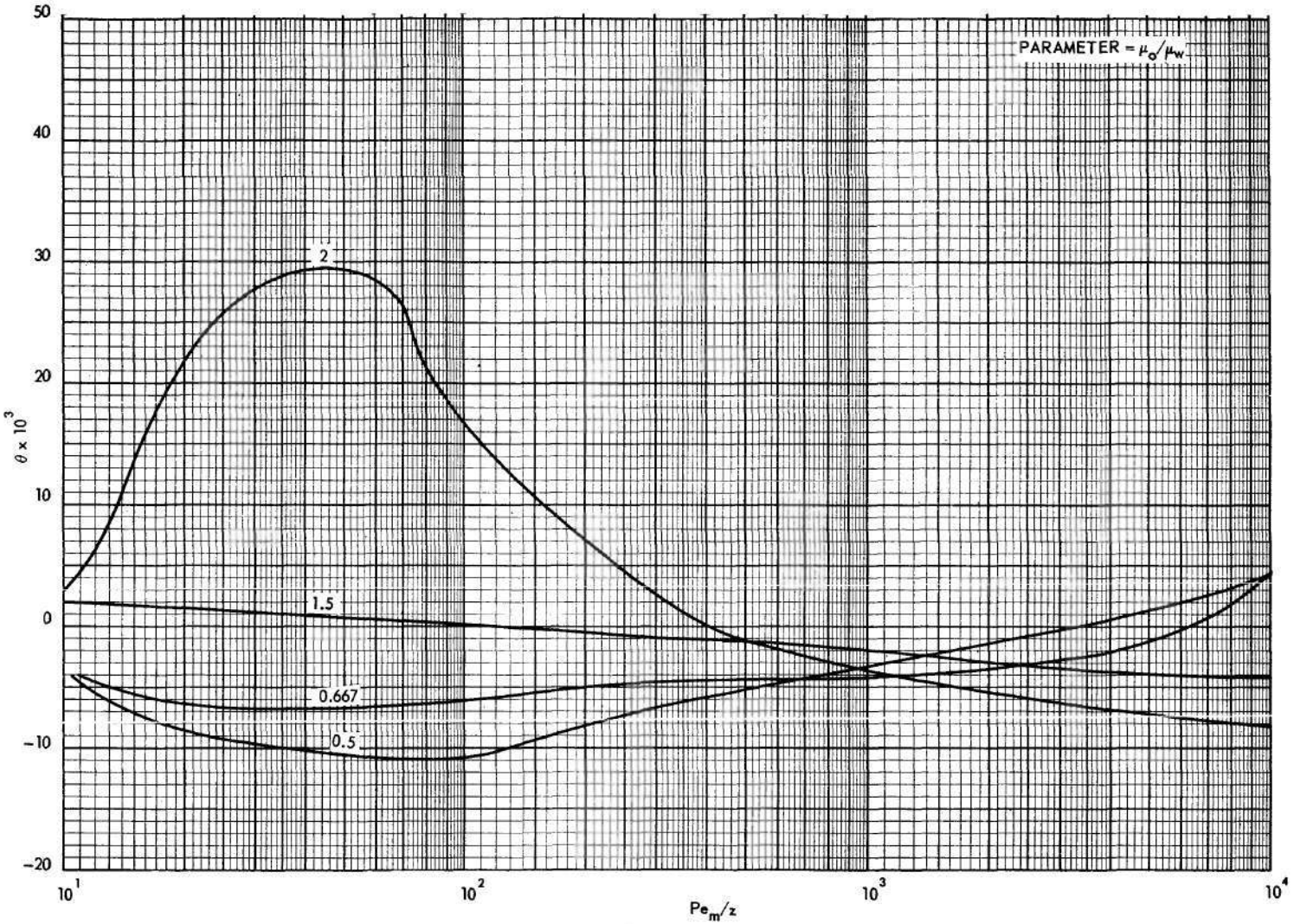

Figure 27. Correction Factor for Effect of Viscosity on Free Convection for Gases with Positive Fc.

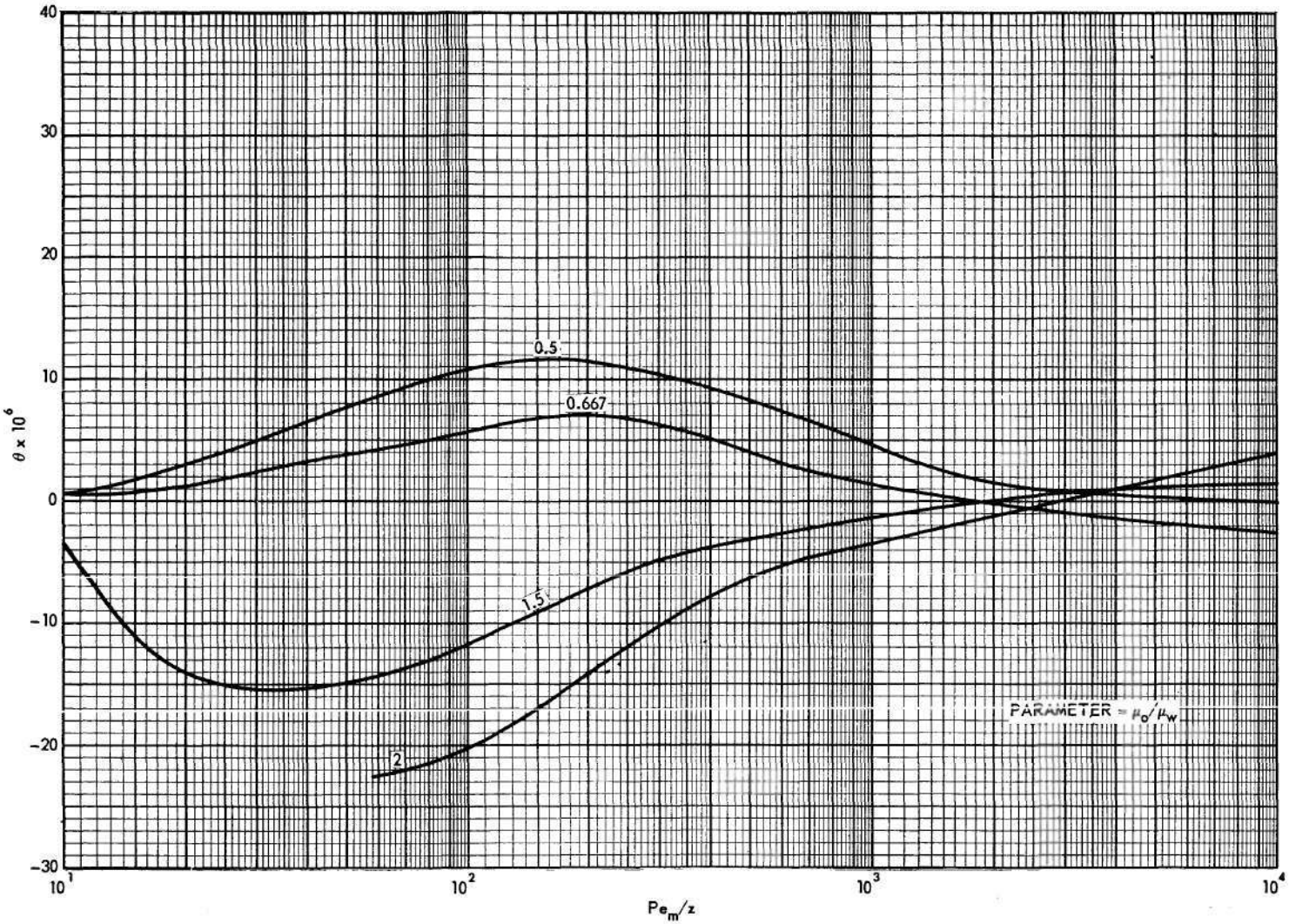

Figure 28. Correction Factor for Effect of Viscosity on Free Convection for Gases with Negative Fc.

on the mean temperature. This figure applies to liquids and gases, heating and cooling, and upflow and downflow. (It is based on cases for which Fc = ± 0.1.) Figure 24 gives a correction factor, ϕ, which accounts for the effect of free convection on T_m in the absence of viscosity variation. This figure likewise applies for liquids and gases, heating and cooling and upflow and downflow. There is also the effect of viscosity on free convection to be taken into account; this is done in Figures 25 through 28. Four separate figures are necessary here because the results for gases differ from those for liquids and those for positive Fc (heating in upflow, cooling in downflow) differ from those with negative Fc (heating in downflow, cooling in upflow).

To compute T_m, one must first read T_m corrected for viscosity alone, T_{mv}, from Figure 23. Next, the correction factor ϕ is read from Figure 24. Finally, the correction factor θ is read from one of the Figures 25-28. From θ, a quantity α is computed according to the formula:

$$\begin{aligned} \alpha &= (Fc)^{1/2}\ \theta,\ Fc > 0 \\ &= (Fc)^{2}\ \theta,\ Fc < 0 \end{aligned} \qquad (16)$$

The mean temperature, T_m, is then calculated from the equation

$$T_m = T_{mv}\ \phi\ (1 + \alpha) \qquad (17)$$

This correlation scheme is sufficiently accurate for practical design calculations (it usually represents the data well within 10 per cent).

Some trial-and-error may be required, since Pe must be evaluated at T_m, but this quantity can usually be estimated with sufficient accuracy that only one trial is necessary.

To illustrate the use of these figures, the following case is considered as an example. Oil A is heated in upflow under conditions such that $\mu_o/\mu_w = 20$, Fc = 800, $T'_o = 540°R$ and $T'_w = 701°R$. The Reynolds number at the entrance to the heating section is 761; the Prandtl number is 800. The mean temperature at z = 512 is desired.

The thermal conductivity of oil A is given by the equation

$$k' = 0.085994 - 2.19474 \times 10^{-5} T', \quad (T' \text{ in } °R) \tag{18}$$

and the heat capacity is given by

$$c'_p = 0.15780 + 5.31579 \times 10^{-4} T', \quad (T' \text{ in } °R) \tag{19}$$

These equations are given in Appendix E. At $T'_o = 540°R$, k' = 0.0741 and $c'_p = 0.445$. As a first trial, assume $T'_m = 560°R$. Here, k' = 0.0737 and $c'_p = 0.455$. For this situation

$$Pe_m/z = \frac{(761)(800)}{(512)} \left(\frac{0.455}{0.445}\right)\left(\frac{0.0741}{0.0737}\right) = 1220 \tag{20}$$

From Figure 23, at $\mu_o/\mu_w = 20$ and $Pe_m/z = 1220$, $T_{mv} = 0.078$. From Figure 24, at Fc = 800 and $Pe_m/z = 1220$, $\phi = 1.8$. From Figure 25, which applies to liquids with Fc > 0, $\theta = 0.0003$. Then

$$\alpha = \theta\,(Fc)^{0.5} = (0.0003)(800)^{0.5} = 0.008 \tag{21}$$

Thus,

$$T_m = T_{mv}\phi\ (1 + \alpha) = (0.078)(1.8)(1.008) = 0.142 \qquad (22)$$

To check our assumed mean temperature, we compute T'_m:

$$T'_m = T'_o + T_m\ (T'_w - T'_o) = 540 + 0.142\ (161) = 563°R \qquad (23)$$

This is so close to the assumed temperature that no further calculations are necessary. In this case the answer is exact since this particular point was used in constructing the curves.

The more exact calculated results are given in Tables 11-24. These results are not tabulated with sufficient frequency that interpolation can be satisfactorily employed; however, the results can be plotted and the curves easily drawn. When entries cease at a value of Pe_m/z much greater than 10, this means that the flow has become unstable (i.e., either the velocity at r = 0 or near the wall has dropped below zero). Even when using the Figures 15-20, these tables should be consulted to determine if there is a likelihood of unstable flow resulting.

Friction factors are much more easily handled because there is little apparent effect of free convection on these quantities. The ratio f/f_w is given in Figure 29 for liquids; the quantity $(f/f_w)(\rho_m/\rho_w)$ is given in Figure 30 for gases. Because these graphs give the friction factors accurately, these quantities are not presented in tables also.

As discussed in Appendix B, the total pressure drop is computed from the relation

$$\Delta p = \frac{1}{Fr}\int_o^z \rho_m dz + (\Delta p)_w \frac{f}{f_w} \qquad (24)$$

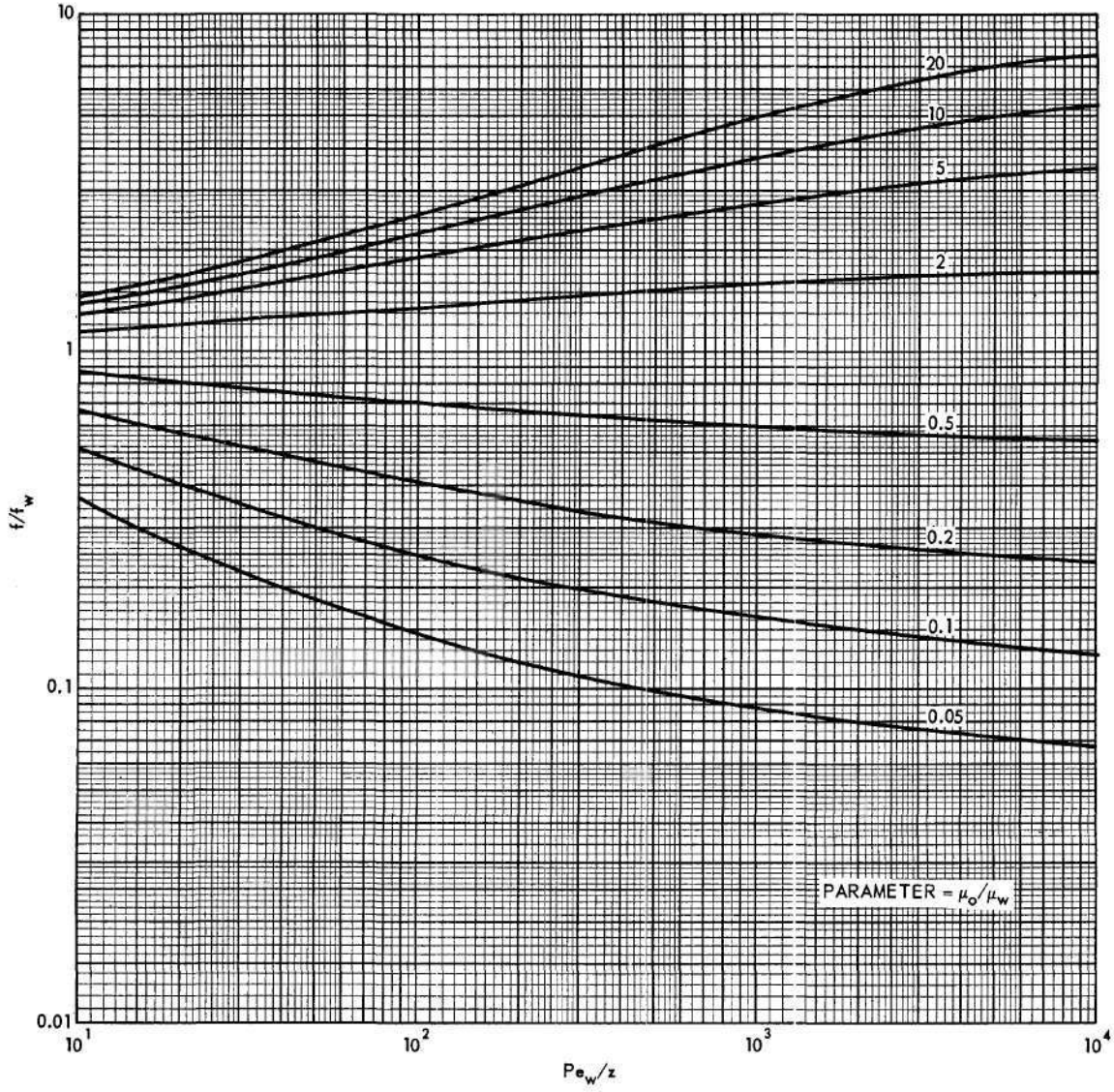

Figure 29. Friction Factors for Liquids.

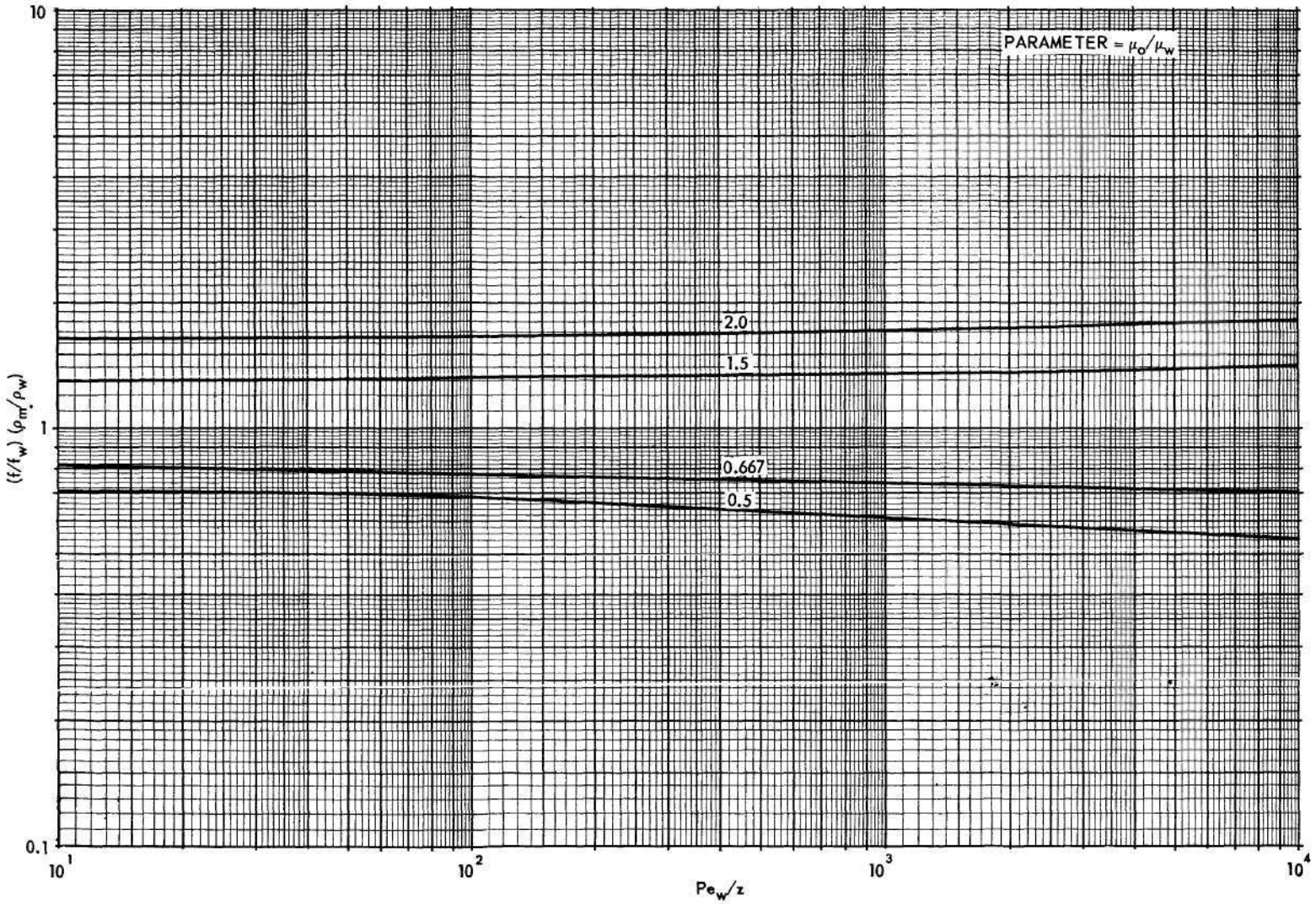

Figure 30. Friction Factors for Gases.

The integral $\int_0^z \rho_m dz$ is evaluated from a knowledge of T_m as a function of z, which is obtainable from Tables 11-24.

The local Nusselt number is tabulated in Tables 11-24. This quantity is of doubtful practical value, but it is included for completeness. In some cases (e.g., Table 21), values of Nu are not tabulated over the entire range of Pe_m/z investigated. This is because, as mentioned previously, accuracy is lost at low values of Pe/z.

The average heat transfer coefficient is often desired; it can be computed easily from knowledge of T_m by the relation:

$$Q = h_m A (\Delta T')_m = w \int_{T'_o}^{T'_m} c'_p dT' \qquad (25)$$

Velocity and temperature profiles are of small value to the designer, but they may be of more value to the researcher. Temperature and velocity profiles are presented in Tables 25-38. The use of these tables is tedious, since both T and u (or $u\rho$) are tabulated as functions of Pe_L/z. While the velocity profiles can be obtained directly, trial-and-error is required to find the temperature profiles.

To illustrate the techniques involved in establishing the temperature profile, consider heating water in upflow under conditions such that $\mu_o/\mu_w = 2$, Fc = 0.1, $T'_o = 540°R$ and $T'_w = 610°R$. The Reynolds number at the entrance to the heating section is 1000; the Prandtl number is 5.89. The temperature at z = 10.72 and r = 0.45 will be found to illustrate the method.

The necessary data are plotted in Figure 13. The heat capacity

of water is constant; the thermal conductivity is given by the equation

$$k' = -0.49958 + 2.58911 \times 10^{-3} T' - 1.875 \times 10^{-6} T'^{2} \quad (T' \text{ in } °R) \quad (26)$$

which is given in Appendix E. At $T'_o = 540°R$, $k' = 0.352$. As a first trial, assume $T' = 580°R$ at the point in question. Here, $k' = 0.371$. Then

$$Pe_L/z = \frac{(1000)(5.89)}{(10.72)} \left(\frac{0.352}{0.371}\right) = 5.22 \times 10^2 \quad (27)$$

From Figure 11,

$$T = 0.590 \quad (28)$$

To check our assumed temperature, we compute T':

$$T' = T'_o + T(T'_w - T'_o) = 540 + 0.590\ (70) = 581°R \quad (29)$$

This is so close to the assumed temperature that no further calculations are necessary.

These tables fail to show very impressively how velocity profiles behave when free convection and variable viscosity effects are important. Since these velocity profiles may be of general interest, typical examples are shown in Figures 31 and 32. Especially interesting is the approach of the center-line velocity to zero when free-convection effects are important in cooling in downflow and heating in upflow. This profile shape is qualitatively confirmed by the experimental data of Watzinger and Johnson (20) and by the visual observations of Scheele et al. (19).

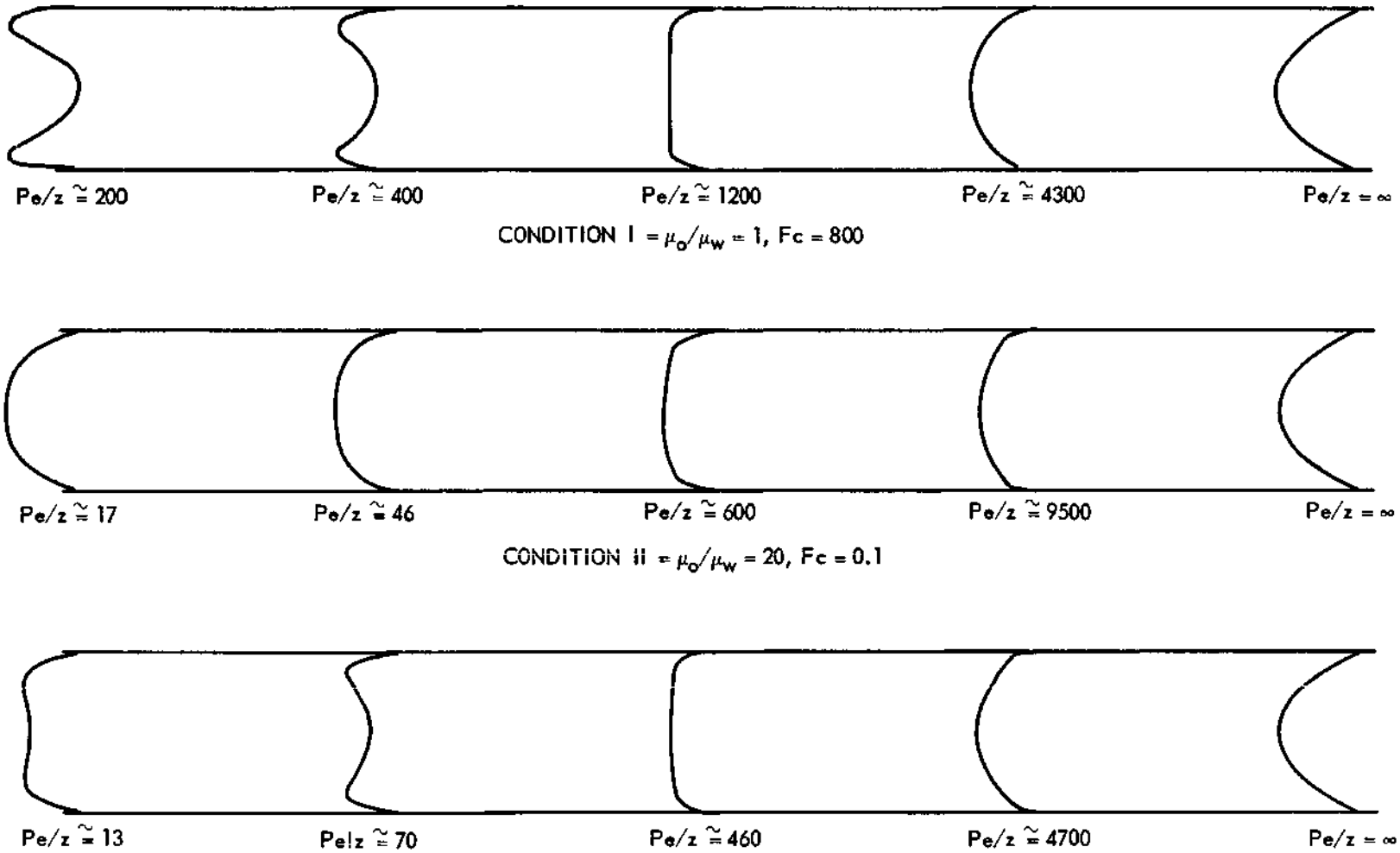

Figure 31. Shapes of Typical Velocity Profiles for Liquids Heated in Upflow.

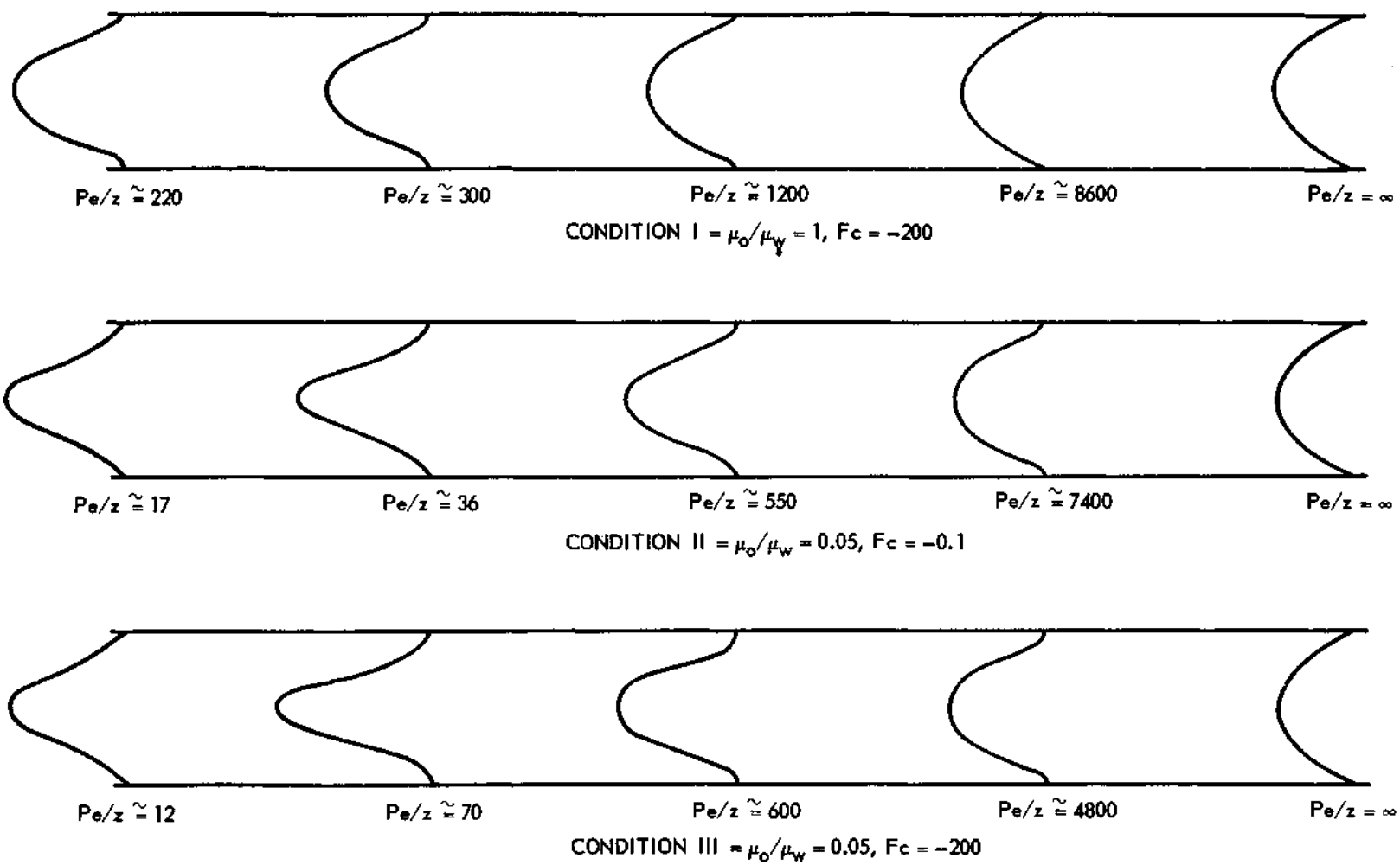

Figure 32. Shapes of Typical Velocity Profiles for Liquids Cooled in Upflow.

It should be pointed out that the results reported here for vertical tubes with free-convection effects unimportant may be used equally well with horizontal tubes. A suggested maximum value of Fc for such purposes is 10.0.

Effect of Pressure on Results for Gases.--The results for gases were obtained using the physical properties of air at one atmosphere pressure. These results should be equally valid at other pressures, except at very high pressures or other unusual conditions under which the gas density obeys the perfect gas laws very poorly.

The question of pressure drop in the gas may appear to complicate the results. However, under most commonly encountered conditions, it seems likely that the pressure drop will be small (less than one pound per square inch), so that there is no need to include the effect of pressure on the density of the gas. Accordingly, in all results for gases reported in this study, the density of the gas was considered to be a function of temperature only. For very small tubes (less than 1/4 inch in diameter), the pressure drop can be significant; in such cases, it will be necessary to modify the numerical scheme and make the necessary calculations on a computer.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

On the basis of the results of this study, the following conclusions were reached:

1. A practical and apparently accurate numerical scheme has been developed to solve the problem of laminar flow in the thermal entrance region of a vertical circular tube with uniform wall temperature for fluids with variable physical properties.

2. A practical scheme for correlating the heat transfer and fluid friction properties of various fluids has been developed. The correlation scheme has been shown to apply to a typical oil whose viscosity may vary by as much as a factor of 20 over the temperature range considered, and to water and gases, whose densities deviate significantly from linearity. It is believed that the results are valid for other substances whose properties are similar to those studied here.

3. The results show that both free convection and variable viscosity effects are quite important, and neither can be arbitrarily neglected in a complete analysis.

4. Variations in heat capacity and thermal conductivity are of some importance, but can be taken into account by evaluating correlating parameters at proper temperatures, rather than having to introduce further correlating parameters.

5. For fluids with $Pr \geq 100$, the results are independent of the initial velocity profile; for fluids with $5 \leq Pr \leq 100$, the results <u>may</u>

be independent of the initial velocity profile. For gases, the initial profile must be parabolic for the results to be useable.

This study suggests that still further research will be fruitful. The following studies are suggested:

1. The problem of this study extended to include developing velocity profile.

2. The problem of this study extended to horizontal tubes, with and without developing velocity profile.

3. Experimental studies in the "unstable" region of flow.

4. Further experimental work with various substances, in both heating and cooling, with choice of conditions based on the results of this study.

It is thought that (2) and (3) are particularly difficult problems, but they are certainly significant ones.

A P P E N D I C E S

APPENDIX A

DERIVATION OF EQUATIONS

Consider the steady-state laminar flow of a Newtonian fluid in a vertical tube. It is assumed that there is axial symmetry, negligible angular velocity, isotropic fluid, negligible bulk viscosity, and that the Peclet number is greater than about ten. Under these conditions, the shear stress components are (22):

$$\tau_{rr} = -2\mu' \frac{\partial v'}{\partial r'} + \frac{2}{3}\mu'\left(\frac{1}{r'}\frac{\partial}{\partial r'}(r'v') + \frac{\partial u'}{\partial z'}\right) \tag{30}$$

$$\tau_{\theta\theta} = -\frac{2\mu'v'}{r'} + \frac{2}{3}\mu'\left(\frac{1}{r'}\frac{\partial}{\partial r'}(r'v') + \frac{\partial u'}{\partial z'}\right) \tag{31}$$

$$\tau_{zz} = -2\mu' \frac{\partial u'}{\partial z'} + \frac{2}{3}\mu'\left(\frac{1}{r'}\frac{\partial}{\partial r'}(r'v') + \frac{\partial u'}{\partial z'}\right) \tag{32}$$

$$\tau_{r\theta} = \tau_{\theta r} = 0 \tag{33}$$

$$\tau_{\theta z} = \tau_{z\theta} = 0 \tag{34}$$

$$\tau_{zr} = \tau_{rz} = -\mu'\left(\frac{\partial u'}{\partial r'} + \frac{\partial v'}{\partial z'}\right) \tag{35}$$

Primed quantities denote variables with dimensions (in contrast with dimensionless variables, which have no primes).

With the introduction of these shear stress components, the equations of motion in the r and z directions (23) become

$$\rho' \left(v' \frac{\partial v'}{\partial r'} + u' \frac{\partial v'}{\partial z'}\right) = - \frac{\partial p'}{\partial r'} - \left[\frac{1}{r'} \frac{\partial}{\partial r'} \left(-2\mu' r' \frac{\partial v'}{\partial r'}\right.\right. \tag{36}$$

$$+ \frac{2}{3} \mu' \frac{\partial}{\partial r'} (r'v') + \frac{2}{3} \mu' r' \frac{\partial u'}{\partial z'}) + \frac{2\mu' v'}{r'^2} - \frac{2}{3} \frac{\mu'}{r'^2} \partial \frac{(r'v')}{\partial r'}$$

$$\left. - \frac{2}{3} \frac{\mu'}{r'} \frac{\partial u'}{\partial z'} - \frac{\partial}{\partial z'} \left(\mu' \frac{\partial u'}{\partial r'} + \mu' \frac{\partial v'}{\partial z'}\right)\right]$$

and

$$\rho' \left(v' \frac{\partial u'}{\partial r'} + u' \frac{\partial u'}{\partial z'}\right) = - \frac{\partial p'}{\partial z'} - \left[\frac{1}{r'} \frac{\partial}{\partial r'} \left(-\mu' r' \left(\frac{\partial u'}{\partial r'} + \frac{\partial v'}{\partial z'}\right)\right)\right. \tag{37}$$

$$\left. + \frac{\partial}{\partial z'} \left(-2\mu' \frac{\partial u'}{\partial z'} + \frac{2}{3} \mu' \left(\frac{1}{r'} \frac{\partial}{\partial r'} (r'v') + \frac{\partial u'}{\partial z'}\right)\right)\right] + \rho' g_z'$$

The continuity equation is (24)

$$\frac{\partial}{\partial r'} (\rho' r' v') + r' \frac{\partial}{\partial z'} (\rho' u') = 0 \tag{38}$$

The energy equation is (25)

$$\rho' c_p' \left(v' \frac{\partial T'}{\partial r'} + u' \frac{\partial T'}{\partial z'}\right) = \frac{1}{r'} \frac{\partial}{\partial r'} \left(r' k' \frac{\partial T'}{\partial r'}\right) + \frac{\partial}{\partial z'} \left(k' \frac{\partial T'}{\partial z'}\right) \tag{39}$$

$$+ \rho' T' \left(\frac{\partial (\frac{1}{\rho'})}{\partial T'}\right)_{\rho'} \left(v' \frac{\partial p'}{\partial r'} + u' \frac{\partial p'}{\partial z'}\right) - \left\{\left[-2\mu' \frac{\partial v'}{\partial r'}\right.\right.$$

$$\left. + \frac{2}{3} \mu' \left(\frac{1}{r'} \frac{\partial}{\partial r'} (r'v') + \frac{\partial u'}{\partial z'}\right)\right] \frac{\partial v'}{\partial r'} + \left[- \frac{2\mu' v'}{r'} + \frac{2}{3} \mu'\right.$$

$$\left. \left(\frac{1}{r'} \frac{\partial}{\partial r'} (r'v') + \frac{\partial u'}{\partial z'}\right)\right] \frac{v'}{r'} + \left[-2\mu' \frac{\partial u'}{\partial z'} + \frac{2}{3} \mu' \left(\frac{1}{r'} \frac{\partial}{\partial r'} (r'v')\right.\right.$$

$$\left.\left. + \frac{\partial u'}{\partial z'}\right)\right] \frac{\partial u'}{\partial z'} - \mu' \left(\frac{\partial u'}{\partial r'} + \frac{\partial v'}{\partial z'}\right)\left(\frac{\partial u'}{\partial r'} + \frac{\partial v'}{\partial z'}\right)\Bigg\}$$

To simplify these equations, non-dimensional quantities, as defined in the Nomenclature section, are introduced. Then a qualitative "order-of-magnitude" analysis is performed. (A similar analysis is given by Schlicting (26).) This procedure is, at best, non-rigorous, but it is an attempt to simplify the equations as systematically as possible. Some authors (14, 16) prefer to simply give the final equations presented here with little comment.

The non-dimensional equations are given below. Given beneath these equations are certain orders-of-magnitude, which are explained in detail following the equations.

The continuity equation becomes

$$\frac{\partial}{\partial r}(\rho v r) + r\frac{\partial}{\partial z}(\rho u) = 0 \tag{40}$$

The equations of motion become

$$\rho r\left(v\frac{\partial v}{\partial r} + u\frac{\partial v}{\partial z}\right) = -r\frac{\partial p}{\partial r} - \frac{1}{Re}\left[\frac{\partial}{\partial r}\left(-2\mu r\frac{\partial v}{\partial r}\right.\right. \tag{41}$$

$$(\epsilon^2 \qquad \epsilon^2) \qquad ? \qquad \epsilon \qquad \left[\left(\quad \epsilon\right.\right.$$

$$\left. + \frac{2}{3}\mu\frac{\partial}{\partial r}(rv) + \frac{2}{3}\mu r\frac{\partial u}{\partial z}\right) + \frac{2\mu v}{r} - \frac{2}{3}\frac{\mu}{r}\frac{\partial}{\partial r}(rv) - \frac{2}{3}\mu\frac{\partial u}{\partial z}$$

$$\epsilon \qquad\qquad \epsilon \quad) \quad \epsilon \qquad\qquad \epsilon \qquad\qquad \epsilon$$

$$\left. - r\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial r} + \mu\frac{\partial v}{\partial z}\right)\right]$$

$$\epsilon \quad (\quad 1 \qquad \epsilon^2 \;)\Big]\Big]$$

and

$$\rho r\left(v\frac{\partial u}{\partial r}+u\frac{\partial u}{\partial z}\right)=-r\frac{\partial p}{\partial z}+\frac{r\rho}{Fr}-\frac{1}{Re}\left[\frac{\partial}{\partial r}\left(-\mu r\left(\frac{\partial u}{\partial r}+\frac{\partial v}{\partial z}\right)\right)\right. \tag{42}$$

$$\left.+\frac{\partial}{\partial z}\left(-2\mu r\frac{\partial u}{\partial z}+\frac{2}{3}\mu\left(\frac{\partial}{\partial r}(rv)+r\frac{\partial u}{\partial z}\right)\right)\right]$$

The energy equation becomes

$$\rho r c_p\left(v\frac{\partial \bar{T}}{\partial r}+u\frac{\partial \bar{T}}{\partial z}\right)=\frac{1}{Pe}\left[\frac{\partial}{\partial r}\left(r\,k\frac{\partial \bar{T}}{\partial r}\right)+r\frac{\partial}{\partial z}\left(k\frac{\partial \bar{T}}{\partial z}\right)\right] \tag{43}$$

$$+\frac{V^2}{\bar{T}_o' c_{p_o}'}\rho r\bar{T}\frac{\partial\left(\frac{1}{\rho}\right)}{\partial \bar{T}}\left(v\frac{\partial p}{\partial r}+u\frac{\partial p}{\partial z}\right)-\frac{\mu_o' V}{D\bar{T}_o' c_{p_o}' \rho_o'}\left\{\left[-2\mu r\frac{\partial v}{\partial r}\right.\right.$$

$$\left.\left.+\frac{2}{3}\mu\left(\frac{\partial}{\partial r}(rv)+r\frac{\partial u}{\partial z}\right)\right]\frac{\partial v}{\partial r}+\left[-2\mu v+\frac{2}{3}\mu\left(\frac{\partial}{\partial r}(rv)\right.\right.\right.$$

$$\left.\left.\left.+r\frac{\partial u}{\partial z}\right)\right]\frac{v}{r}+\left[-2\mu r\frac{\partial u}{\partial z}+\frac{2}{3}\mu\left(\frac{\partial}{\partial r}(rv)+r\frac{\partial u}{\partial z}\right)\right]\frac{\partial u}{\partial z}\right.$$

$$\left.-\mu r\left(\frac{\partial u}{\partial r}+\frac{\partial v}{\partial z}\right)^2\right\}$$

It is postulated that the order of magnitude of several quantities be unity <u>at most</u>; i.e., the values of these variables do not differ from the reference values by a large amount. The notation O(x) means

"of order no greater than x." It is assumed that

$$\begin{aligned} u &= O(1) \\ \mu &= O(1) \\ \overline{T} &= O(1) \\ c_p &= O(1) \\ \rho &= O(1) \\ v &= O(\epsilon) \\ v/r &= O(\epsilon) \end{aligned} \tag{44}$$

where

$$O(\epsilon) << O(1)$$

It is further assumed that differentiation with respect to r does not change the order of magnitude of a quantity; i.e.,

$$O\left(\frac{\partial y}{\partial r}\right) = O(y) \tag{45}$$

The reasoning is this: the variables of interest, u, $\overline{T}$, and v change from their minimum to their maximum values in a distance of r = 1/2 or less. For example, u changes in value from 0 to 2, for example, as r changes from 1/2 to 0; thus

$$\left(\frac{\partial u}{\partial r}\right)_{avg} = \frac{\Delta u}{\Delta r} = \frac{2}{-1/2} = -4 \sim O(1) \tag{46}$$

On the other hand, none of these quantities changes rapidly in the z-direction (for example, except for Pe quite small, it takes the fluid a considerable distance to approach the wall temperature), so it

is assumed that differentiation with respect to z decreases the order of magnitude of a quantity:

$$O\left(\frac{\partial y}{\partial z}\right) = O(\epsilon y) \tag{47}$$

From these postulates and their consequences, the order of magnitude of the various terms in the equations of motion and energy equation can be qualitatively estimated. These orders of magnitude are indicated beneath the various terms.

It is assumed that $1/Re$ is of order ϵ.

Note that, in equation (41), all terms are of order ϵ^2 or less; thus,

$$\frac{\partial p}{\partial r} \sim O(\epsilon^2) \simeq O \tag{48}$$

Accordingly, to a good approximation, the pressure is a function of z alone.

In equation (42), if the pressure gradient is to be as important as the other terms, $\frac{\partial p}{\partial z} \sim O(\epsilon)$. There is certainly no justification for neglecting this term. This is likewise true for the term ρ/Fr.

In the energy equation, it is noted that the term $V^2/T'_o c'_{p_o}$ is closely related to the Mach number, which, for a perfect gas, is $V^2/(\gamma-1)T'c'_p$. For laminar flow in tubes at temperatures and tube diameters ordinarily encountered, the Mach number is quite small. Assuming $\rho\overline{T}\,\frac{\partial(\frac{1}{\rho})}{\partial\overline{T}} = O(1)^*$ the second term on the right side of the equation is

*For a perfect gas, this term is unity; for a completely incompressible liquid, it is zero.

negligible. The group $\mu_o' V/DT_o' c_{p_o}' \rho_o' = \frac{1}{Re} \frac{V^2}{c_{p_o}' T_o'}$ is likewise negligible.

After eliminating the terms of small order of magnitude, the basic differential equations become

$$\frac{\partial}{\partial r}(\rho v r) + r \frac{\partial}{\partial z}(\rho u) = 0 \tag{49}$$

$$\rho\left(v \frac{\partial u}{\partial r} + u \frac{\partial u}{\partial z}\right) = -\frac{dp}{dz} + \frac{\rho}{Fr} + \frac{1}{Re}\left[\frac{1}{r} \frac{\partial}{\partial r}\left(\mu r \frac{\partial u}{\partial r}\right)\right] \tag{50}$$

$$\rho c_p \left(v \frac{\partial \bar{T}}{\partial r} + u \frac{\partial \bar{T}}{\partial z}\right) = \frac{1}{rPe} \frac{\partial}{\partial r}\left(rk \frac{\partial \bar{T}}{\partial r}\right) \tag{51}$$

In equation (51), it is convenient to introduce the variable $T = \bar{T} - T_o'/(T_w' - T_o') = (T' - T_o')/(T_w' - T_o')$. With this change in variables, the equation becomes

$$\rho c_p \left(v \frac{\partial T}{\partial r} + u \frac{\partial T}{\partial z}\right) = \frac{1}{rPe} \frac{\partial}{\partial r}\left(rk \frac{\partial T}{\partial r}\right) \tag{52}$$

To facilitate solution of the system of equations (49), (50), and (52), flows with velocity profiles sufficiently developed that v and $\frac{\partial u}{\partial z}$ are <u>very</u> small (of order ϵ^2, for example) will be considered, as was done by Pigford (11). (It is difficult to justify this step except by the fact that experimental data agree well with computed results.) With this simplification, the equations of motion and energy become

$$G = \frac{dp}{dz} = \frac{\rho}{Fr} + \frac{1}{rPe} \frac{\partial}{\partial r}\left(\mu r \frac{\partial u}{\partial z}\right) \tag{53}$$

and

$$\rho c_p u \frac{\partial T}{\partial z} = \frac{1}{rPe} \frac{\partial}{\partial r}\left(rk \frac{\partial T}{\partial r}\right) \tag{54}$$

The result

$$u = \frac{Re}{Fr} \int_r^{1/2} \frac{1}{\mu r} \left(\int_o^r s\rho ds \right) dr - \frac{ReG}{2} \int_r^{1/2} \frac{r}{\mu} dr \tag{55}$$

is obtained by integrating the equation of motion twice. Integration of the continuity equation yields

$$\int_o^{1/2} r \frac{\partial}{\partial z} (\rho u) dr = - \int_o^{1/2} \frac{\partial}{\partial r} (\rho v r) dr = 0 \tag{56}$$

But

$$\int_o^{1/2} r \frac{\partial}{\partial z} (\rho u) dr = \int_o^{1/2} \frac{\partial}{\partial z} (\rho u r) dr = \frac{d}{dz} \left[\int_o^{1/2} \rho u r \, dr \right] = 0 \tag{57}$$

The implication is

$$\int_o^{1/2} \rho u r dr = \text{constant} = \int_o^{1/2} \rho u r dr \Big|_{z=0} = 1/8 \tag{58}$$

After combining the integrated continuity and momentum equations, the equation

$$G = \frac{\frac{Re}{Fr} \left\{ \int_o^{1/2} \rho r \left[\int_r^{1/2} \frac{1}{y\mu} \left(\int_o^y s\rho ds \right) dy \right] dr \right\} - 1/8}{\frac{Re}{2} \int_o^{1/2} \rho r \left(\int_r^{1/2} \frac{y}{\mu} dy \right) dr} \tag{59}$$

is obtained.

APPENDIX B

DEVELOPMENT OF NUMERICAL SCHEME

Approximations for Derivatives.--Before deriving useful expressions for approximating derivatives, it is desirable to explain certain terminology and notation.

The region of interest in the problem treated is some portion of the (r,z) plane; for example, the region can be considered to be the semi-infinite strip of those points (r,z) such that $0 \leq r \leq 1/2$ and $z \geq 0$. Let $\Delta r = 1/N$, where N is a positive integer, and let Δz be a positive number. We shall be interested in the values of some function, say $h(r,z)$, at the points $(i\Delta r, n\Delta z)$, where i and n are integers and where $1 \leq i \leq N + 1$. The notation h_{in} will be used to denote the value of h at the point $((i-1)\ \Delta r, n\Delta z)$:

$$h_{in} = h((i-1)\ \Delta r, n\Delta z) \tag{60}$$

When it is said that a certain expression is "correct to the second-order" in Δz, the implication is that the truncation error made in approximating a quantity by this expression is no greater than some constant multiplied by $(\Delta z)^2$. This notion will be clarified in the development below.

There are many ways of approximating the same derivative. The particular choice depends on the accuracy desired and whether use of the approximation will lead to a finite difference scheme, the solution of which will converge to the solution of the original differential

equation and which can reasonably be expected to be stable (even if this stability and convergence cannot rigorously be demonstrated, as is the case with the numerical scheme developed in this study). In addition, certain types of approximations greatly facilitate the solution of the resulting difference scheme. Accordingly, if some of the approximations below appear mysterious, it should be remembered that they were made with these criteria in mind.

It is first desired to develop an approximation to $(\frac{\partial h}{\partial r})_{in}$ which is correct to the second order in Δr. Note that, by a Taylor series expansion,

$$h_{i+1,n} = h_{in} + \Delta r \left(\frac{\partial h}{\partial r}\right)_{in} + \frac{(\Delta r)^2}{2} \left(\frac{\partial^2 h}{\partial r^2}\right)_{in} + c_1(\Delta r)^3 \tag{61}$$

and

$$h_{i-1,n} = h_{in} - \Delta r \left(\frac{\partial h}{\partial r}\right)_{in} + \frac{(\Delta r)^2}{2} \left(\frac{\partial^2 h}{\partial r^2}\right)_{in} + c_2(\Delta r)^3 \tag{62}$$

The constants c_i are shorthand notation used to describe the error term in truncating a Taylor series after the first few terms. For example, the term $c_1(\Delta r)^3$ in equation (61) means $\frac{(\Delta r)^3}{3!} \frac{\partial^3 h}{\partial r^3} (\xi, r\Delta z)$ where $(i-1)\Delta r \leq \xi \leq i\Delta r$.

After combining, one obtains

$$\left(\frac{\partial h}{\partial r}\right)_{in} = \frac{(h_{i+1,n} - h_{i-1,n})}{2\Delta r} + c_3(\Delta r)^2 \tag{63}$$

Similarly,

$$\left(\frac{\partial h}{\partial z}\right)_{in} = \frac{(h_{i,n+1} - h_{i,n-1})}{2\Delta z} + c_4 (\Delta z)^2 \tag{64}$$

We now seek an approximation to $\left(\frac{\partial^2 h}{\partial r^2}\right)_{in}$. Note that

$$h_{i+1,n} = h_{in} + \Delta r \left(\frac{\partial h}{\partial r}\right)_{in} + \frac{(\Delta r)^2}{2} \left(\frac{\partial^2 h}{\partial r^2}\right)_{in} + \frac{(\Delta r)^3}{6} \left(\frac{\partial^3 h}{\partial r^3}\right)_{in} + c_5 (\Delta r)^4 \tag{65}$$

and

$$h_{i-1,n} = h_{in} - \Delta r \left(\frac{\partial h}{\partial r}\right)_{in} + \frac{(\Delta r)^2}{2} \left(\frac{\partial^2 h}{\partial r^2}\right)_{in} - \frac{(\Delta r)^3}{6} \left(\frac{\partial^3 h}{\partial r^3}\right)_{in} + c_6 (\Delta r)^4 \tag{66}$$

Combining,

$$\left(\frac{\partial^2 h}{\partial r^2}\right)_{in} = \frac{h_{i+1,n} - 2h_{in} + h_{i-1,n}}{(\Delta r)^2} + c_7 (\Delta r)^2 \tag{67}$$

Also, note that

$$\left(\frac{\partial^2 h}{\partial r^2}\right)_{i,n+1} = \left(\frac{\partial^2 h}{\partial r^2}\right)_{in} + \Delta z \left(\frac{\partial^3 h}{\partial z \partial r^2}\right)_{in} + c_8 (\Delta z)^2 \tag{68}$$

and

$$\left(\frac{\partial^2 h}{\partial r^2}\right)_{i,n-1} = \left(\frac{\partial^2 h}{\partial r^2}\right)_{in} - \Delta z \left(\frac{\partial^3 h}{\partial z \partial r^2}\right)_{in} + c_9 (\Delta z)^2 \tag{69}$$

Adding and rearranging,

$$\left(\frac{\partial^2 h}{\partial r^2}\right)_{in} = \frac{1}{2}\left[\left(\frac{\partial^2 h}{\partial r^2}\right)_{i,n+1} + \left(\frac{\partial^2 h}{\partial r^2}\right)_{i,n-1}\right] + c_{10} (\Delta z)^2 \tag{70}$$

Combining the results expressed in equations (67) and (70),

$$\left(\frac{\partial^2 h}{\partial r^2}\right)_{in} = \frac{1}{2(\Delta r)^2}\Big[h_{i+1,n+1} - 2h_{i,n+1} + h_{i-1,n+1} + h_{i+1,n-1} - 2h_{i,n-1} + h_{i-1,n-1}\Big] + c_{11}(\Delta r)^2 + c_{12}(\Delta z)^2 \tag{71}$$

Thus, we have developed an approximation to $\left(\frac{\partial^2 h}{\partial r^2}\right)_{in}$ which is accurate to the second order in Δr and Δz and which, it turns out, is valuable for reducing the non-linear energy differential equation to a linear difference equation.

It is necessary to find an approximation to $\frac{\partial h}{\partial r}$ at $r = 0$ for use with the boundary conditions. By definition, $\left(\frac{\partial h}{\partial r}\right)_{r=0,n} \equiv \left(\frac{\partial h}{\partial r}\right)_{1,n}$. Note that

$$h_{2,n} = h_{1,n} + \Delta r \left(\frac{\partial h}{\partial r}\right)_{1,n} + \frac{(\Delta r)^2}{2}\left(\frac{\partial^2 h}{\partial r^2}\right)_{1,n} + c_{13}(\Delta r)^3 \tag{72}$$

and

$$h_{3,n} = h_{1,n} + 2\Delta r \left(\frac{\partial h}{\partial r}\right)_{1,n} + 2(\Delta r)^2 \left(\frac{\partial^2 h}{\partial r^2}\right)_{1,n} + c_{14}(\Delta r)^3 \tag{73}$$

Combining,

$$\left(\frac{\partial h}{\partial r}\right)_{1,n} = -\frac{3h_{1,n} + 4rh_{2,n} - h_{3,n}}{2\Delta r} + c_{15}(\Delta r)^2 \tag{74}$$

In a similar manner, we obtain an approximation to the derivative at $r = 1/2$; i.e., at $i = N + 1$. The result is

$$\left(\frac{\partial h}{\partial r}\right)_{N+1,n} = \frac{3h_{N+1,n} - 4h_{N,n} + h_{N-1,n}}{2\Delta r} + c_{16}(\Delta r)^2 \tag{75}$$

Finally, we seek an approximation to $\left(\frac{\partial h}{\partial z}\right)_{in}$ which is valid to only the first order in Δz. Note that

$$h_{i,n+1} = h_{in} + \Delta z \left(\frac{\partial h}{\partial z}\right)_{in} + c_{17}(\Delta z)^2 \tag{76}$$

Rearranging,

$$\left(\frac{\partial h}{\partial z}\right)_{in} = \frac{(h_{i,n+1} - h_{in})}{\Delta z} + c_{17}(\Delta z) \tag{77}$$

Energy Equation.--Using equations (63), (64), and (71) in the energy equation, a finite difference approximation accurate to the second order in Δr and Δz is obtained:

$$T_{i-1,n+1} - \left(2 + \frac{Pe\rho_{in} c_{p_{in}} u_{in}(\Delta r)^2}{k_{in}\Delta z}\right) T_{i,n+1} + T_{i+1,n+1} \tag{78}$$

$$= -T_{i-1,n-1} - T_{i+1,n-1} + \left(2 - \frac{Pe\ \rho_{in} c_{p_{in}}\ u_{in}(\Delta r)^2}{k_{in}\Delta z}\right) T_{i,n-1}$$

$$-\Delta r\,(T_{i+1,n} - T_{i-1,n}) \left(\frac{1}{r_i} + \frac{k_{i+1,n} - k_{i-1,n}}{2k_{in}\Delta r}\right)$$

From equation (74), a suitable approximation to the boundary condition (6b) is seen to be

$$3T_{1,n+1} - 4T_{2,n+1} + T_{3,n+1} = 0 \tag{79}$$

Accordingly, given the values of T and u at $z = (n-1)\Delta z$ and

$z = n\Delta z$, the values of T at $z = (n+1)\Delta z$ can be calculated by solving the system of linear equations

$$3T_{1,n+1} - 4T_{2,n+1} + T_{3,n+1} = 0 \tag{80a}$$

$$T_{i-1,n+1} + B_i T_{i,n+1} + T_{i+1,n+1} = D_i, \quad i=2,\ldots,N \tag{80b}$$

$$T_{N+1,n+1} = 1 \tag{80c}$$

To solve this system more readily, it is necessary to eliminate $T_{3,n+1}$ from the equations

$$3T_{1,n+1} - 4T_{2,n+1} + T_{3,n+1} = 0 \tag{81}$$

$$T_{1,n+1} + B_2 T_{2,n+1} + T_{3,n+1} = D_2 \tag{82}$$

The result is

$$T_{1,n+1} - (2 + B_2/2)T_{2,n+1} = -D_2/2 \tag{83}$$

Thus, the system to be solved is

$$T_{1,n+1} - (2 + B_2/2)T_{2,n+1} = -D_2/2 \tag{84a}$$

$$T_{i-1,n+1} + B_i T_{i,n+1} + T_{i+1,n+1} = D_i, \quad i=2,\ldots,N \tag{84b}$$

$$T_{N+1,n+1} = 1 \tag{84c}$$

Using the well-known Gaussian Elimination Scheme for a "tridiagonal" system like the above (27), the equations can be readily solved. Consider the system

$$b_1 x_1 + c_1 x_2 = d_1 \tag{85}$$

$$a_2 x_1 + b_2 x_2 + c_2 x_3 = d_2$$

$$\cdots$$

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i$$

$$\cdots$$

$$a_n x_{n-1} + b_n x_n = d_n$$

It can be shown that a computational algorithm for solving the system is

$$\beta_1 = b_1 \tag{86a}$$

$$\beta_i = b_i - a_i c_{i-1}/\beta_{i-1}, \quad i=2,\ldots,n \tag{86b}$$

$$\gamma_1 = d_1/\beta_1 \tag{87a}$$

$$\gamma_i = (d_i - a_i \gamma_{i-1})/\beta_i, \quad i = 2,\ldots,n \tag{87b}$$

$$x_n = \gamma_n \tag{88a}$$

$$x_i = \gamma_i - c_i x_{i+1}/\beta_i, \quad i = n-1,\ldots,1 \tag{88b}$$

This algorithm was used in the numerical solution of the energy equation.

The questions of stability and convergence of the numerical scheme described above are treated in Appendix D.

A complication arises in using the numerical scheme described above in solving the energy equation. To use the scheme to predict values of temperature at level n+1, one must know values of temperature at levels n and n-1. Accordingly, the method cannot be used to predict

the temperature at $z = \Delta z$ since the temperature is known only at $z = 0$; i.e., at only one previous level instead of the necessary two.

To overcome this difficulty, an explicit scheme was used to start the solution. Such a scheme has the disadvantage of less accuracy (of order $(\Delta r)^2$ and (Δz)) and instability for all but small values of Δz. The problem of lack of accuracy is overcome by choosing the initial Δz to be very small. Later, Δz is increased when using the more accurate scheme in regions where the solution is changing slowly. Likewise, the stability requirement is met by the small initial choice of Δz.

Using equations (63), (64), and (77) in the energy equation to approximate the various derivatives, the resulting finite difference approximation is

$$T_{i,n+1} = T_{in} + \frac{(\Delta z)k_{in}}{(\Delta r)u_{in}\rho_{in}c_{p_{in}}Pe}\left[\left(\frac{1}{r_i} + \frac{k_{i+1,n}-k_{i-1,n}}{2k_{in}\Delta r}\right)\right. \tag{89}$$

$$\left.\left(\frac{T_{i+1,n}-T_{i-1,n}}{2}\right) + \frac{1}{\Delta r}\ (T_{i+1,n}-2T_{in} + T_{i-1,n})\right]$$

This equation is to be used only where $n = 1$. Further, $T_{i,1} \equiv 0$ except for the wall $(i = N + 1)$; here, by choice, $T_{N+1,1} = 1/2$. The equality $T_{i,1} = 0$ implies that all the physical properties are unity. One problem arises: it is necessary to evaluate $k_{N+1,1}$--should this be done at $T_{N+1,1} = 1/2$? It was found that this is not satisfactory, since the difference $(k_{N+1,1} - k_{N-1,1})/\Delta r$, would then be unrealistically large. (Since this difference is related to $\left(\frac{\partial k}{\partial r}\right)_{N,1}$, it is intuitive that the difference should, in fact, equal zero.)

With these facts in mind, it is seen that the explicit scheme reduces to the simple result

$$T_{i,2} = T_{i,1} = 0, \quad i = 1, \ldots, N-1 \tag{90a}$$

$$T_{N,2} = \frac{\Delta z}{4Pe\ u_{N,1}\ (\Delta r)^2} \left(\frac{1}{(N-1)} + 2 \right) \tag{90b}$$

Some remarks about the stability and convergence of the explicit scheme are made in Appendix D.

Equation of Motion.--The problem of evaluating the pressure gradient and velocity profile is essentially a problem in evaluating multiple integrals numerically. There is one very important point, however, that may easily be overlooked. In the early stages of the solution, the temperature gradient is very large near the wall of the tube. Since, for most liquids, the viscosity is very dependent on temperature, this means that there can be a very significant variation in viscosity from point to point near the wall. To achieve as much accuracy as possible in evaluating integrals involving viscosity (such as $\int_r^{1/2} (y/\mu)dy$), and to keep the number of grid points sufficiently small that excessive computer time is not required, it was believed to be wise to devise high-order accurate numerical integration formulas for use near the tube wall.

It was decided that, at the wall, a five-point quadrature formula would be used; thus, a fourth degree polynomial would be fitted through the first five points and integrated from $r = (N + 1)\Delta r$ to $r = N\Delta r$. Accordingly, the required form is

$$\int_{x_3}^{x_4} y dx \simeq A_o y_o + A_1 y_1 + A_2 y_2 + A_3 y_3 + A_4 y_4 \tag{91}$$

The coefficients A_0 through A_4 were found by the method of undetermined coefficients (28); i.e., by solving the system of equations resulting when the requirement is made that the formula be exact for $y = 1$, $y = (x_4 - x_3)$, $y = (x_4 - x_3)^2$, $y = (x_4 - x_3)^3$, and $y = (x_4 - x_3)^4$. The result is

$$\int_{x_3}^{x_4} y\,dx \simeq \frac{\Delta x}{720} (-19y_0 + 106y_1 - 264y_2 + 646y_3 + 251y_4) \qquad (92)$$

It was necessary to develop another quadrature formula for the interval one step removed from the wall. For the remaining steps (an even number), Simpson's rule could conveniently be employed.

For the interval one step removed from the wall, the desired quadrature formula is of the form

$$\int_{x_2}^{x_3} y\,dx \simeq B_0 y_0 + B_1 y_1 + B_2 y_2 + B_3 y_3 \qquad (93)$$

Again using the method of undetermined coefficients, the result was found to be

$$\int_{x_2}^{x_3} y\,dx \simeq \frac{\Delta x}{24} (y_0 - 5y_1 + 19y_2 + 9y_3) \qquad (94)$$

Simpson's rule, used for the remainder of the quadratures, is

$$\int_{x_0}^{x_2} y\,dx \simeq \frac{\Delta x}{3} (y_0 + 4y_1 + y_2) \qquad (95)$$

One further quadrature formula, given by Milne (29), is useful:

$$\int_{x_o}^{x_1} y\,dx \cong \frac{\Delta x}{24}\,(-y_{-1} + 13y_o + 13y_1 - y_2) \tag{96}$$

Applying this formula in a region of symmetry (as is done below), where $y_1 = -y_{-1}$, and where $y_o = 0$, gives

$$\int_{x_o}^{x_1} y\,dx \cong \frac{\Delta x}{24}\,(14y_1 - y_2) \tag{97}$$

To apply these results to the present problem, we first define the integrals

$$P1 = \int_o^r \rho r\,dr \tag{98}$$

$$P2 = \int_r^{1/2} \frac{r}{\mu}\,dr \tag{99}$$

$$P3 = \int_r^{1/2} \frac{P1}{r\mu}\,dr \tag{100}$$

With these definitions, the velocity and pressure gradient equations become

$$G = \frac{\frac{Re}{Fr}\left(\int_o^{1/2} \rho r(P3)dr\right) - 1/8}{\frac{Re}{2}\int_o^{1/2} \rho r(P2)dr} \tag{101}$$

and

$$u = \frac{Re}{Fr}\ (P3) - \frac{ReG}{2}\ (P2) \tag{102}$$

Applying the quadrature formulas developed above, we obtain the computational algorithm

$$Pl_1 = P2_{N+1} = P3_{N+1} = 0 \tag{103a}$$

$$Pl_2 = \frac{\Delta r}{24}\left[14(\Delta r)\rho_2 - 2(\Delta r)\rho_3\right] \tag{103b}$$

$$Pl_i = Pl_{i-2} + \frac{\Delta r}{3}\ (r_i\rho_i + 4r_{i-1}\rho_{i-1} + r_{i-2}\rho_{i-2}) \tag{103c}$$

$$i = 3,4,\ldots,N+1$$

$$P2_N = \frac{\Delta r}{720}\left[-\frac{19r_{N-3}}{\mu_{N-3}} + 106\,\frac{r_{N-2}}{\mu_{N-2}} - 264\,\frac{r_{N-1}}{\mu_{N-1}} + 646\,\frac{r_N}{\mu_N} + 251\,\frac{r_{N+1}}{\mu_{N+1}}\right] \tag{103d}$$

$$P3_N = \frac{\Delta r}{720}\left[-\frac{19\ Pl_{N-3}}{r_{N-3}\mu_{N-3}} + \frac{106\ Pl_{N-2}}{r_{N-2}\mu_{N-2}} - \frac{264\ Pl_{N-1}}{r_{N-1}\mu_{N-1}} + \frac{646\ Pl_N}{r_N\mu_N} + \frac{251\ Pl_{N+1}}{r_{N+1}\mu_{N+1}}\right] \tag{103e}$$

$$P2_{N-1} = P2_N + \frac{\Delta r}{24}\left[\frac{r_{N-3}}{\mu_{N-3}} - 5\,\frac{r_{N-2}}{\mu_{N-2}} + 19\,\frac{r_{N-1}}{\mu_{N-1}} + 9\,\frac{r_N}{\mu_N}\right] \tag{103f}$$

$$P3_{N-1} = P3_N + \frac{\Delta r}{24}\left[\frac{Pl_{N-3}}{r_{N-3}\mu_{N-3}} - 5\,\frac{Pl_{N-2}}{r_{N-2}\mu_{N-2}} + 19\,\frac{Pl_{N-1}}{r_{N-1}\mu_{N-1}} + 9\,\frac{Pl_N}{r_N\mu_N}\right] \tag{103g}$$

$$P2_i = P2_{i+2} + \frac{\Delta r}{3}\left[\frac{r_{i+2}}{\mu_{i+2}} + 4\,\frac{r_{i+1}}{\mu_{i+1}} + \frac{r_i}{\mu_i}\right], \quad i=N-2,N-3,\ldots,2 \tag{103h}$$

$$P3_i = P3_{i+2} + \frac{\Delta r}{3}\left[\frac{Pl_{i+2}}{r_{i+2}\mu_{i+2}} + 4\,\frac{Pl_{i+1}}{r_{i+1}\mu_{i+1}} + \frac{Pl_i}{r_i\mu_i}\right], \tag{103i}$$

$$i=N-2,N-3,\ldots,2$$

The integrals $\int_0^{1/2} (\rho r(P3))dr$ and $\int_0^{1/2} (\rho r(P2))dr$ were integrated using a straight-forward application of Simpson's rule.

Radial Velocity.--Although the radial velocity was not used in the computations, it is desirable to attempt to calculate this quantity to get some feeling for its magnitude.

Consider the continuity equation

$$\frac{\partial}{\partial r}(\rho v r) + r\,\frac{\partial}{\partial z}(\rho u) = 0 \tag{104}$$

For convenience, let $\bar{v} = \rho v r$; then we have

$$\frac{\partial \bar{v}}{\partial r} = -r\,\frac{\partial}{\partial z}(\rho u) \tag{105}$$

A numerical solution is obtained by a simple trapezoidal rule integration:

$$y_{n+1} = y_n + \frac{\Delta x}{2}(y'_{n+1} + y'_n) \tag{106}$$

Applied here, this method becomes

$$\bar{v}_{i,n+1} = \bar{v}_{i+1,n+1} + \frac{\Delta r}{2} \left(r_{i+1} \frac{\partial}{\partial z} (\rho u)_{i+1,n+1} + r_i \frac{\partial}{\partial z} (\rho u)_{i,n+1} \right) \tag{107}$$

To approximate the axial velocity derivatives, we use the approximation

$$\frac{\partial}{\partial z} (\rho u)_{in} = \frac{(\rho u)_{in} - (\rho u)_{i,n-1}}{\Delta z} \tag{108}$$

The final computational scheme becomes (using the boundary condition $\bar{v}_{N+1,n+1} = 0$):

$$\bar{v}_{N+1,n+1} = 0 \tag{109a}$$

$$\bar{v}_{i,n+1} = \bar{v}_{i+1,n+1} + \frac{\Delta r}{2\Delta z} \left[r_{i+1} \left(\rho_{i+1,n+1} u_{i+1,n+1} - \rho_{i+1,n} u_{i+1,n} \right) + r_i \left(\rho_{i,n+1} u_{i,n+1} - \rho_{in} u_{in} \right) \right] \tag{109b}$$

It should be pointed out that a high degree of accuracy cannot be expected in $\bar{v}$. The numerical schemes used to calculate u and T can be expected to give solutions which converge to the solution of the original differential equations. However, no such convergence can be expected for derivatives of the solution (such as $\frac{\partial(\rho u)}{\partial z}$)--particularly in cases such as the present, where the values of ρu at successive steps up the tube are very nearly equal. Accordingly, only qualitative results can be expected.

Mean Temperature.--The mean temperature T'_m is defined as that temperature

which is related to the total rate of heat transfer by the equation

$$w \int_{T'_o}^{T'_m} c'_p dT' = Q \tag{110}$$

To find T'_m, an expression must be developed for Q, the rate of heat transfer.

Consider the first law of thermodynamics written for a "control volume"; i.e., an element of volume fixed in space (here, a length z' of tube) (30):

$$Q = \oint_{C.V.} (H' + \frac{\vec{V}\cdot\vec{V}}{2} + g_z z')\rho'\vec{V}\cdot d\vec{A} \tag{111}$$

For the present case, this reduces to

$$Q = \Delta\left\{2\pi \int_o^R \rho' u' r' (H' + \frac{u'^2}{2} + g z')dr\right\} = w \int_{T'_o}^{T'_m} c'_p dT' \tag{112}$$

Reducing to non-dimensional form and simplifying, the result is

$$\int_1^{\overline{T}_m} c_p d\overline{T} = \Delta\left\{8 \int_o^{1/2} H\rho u r dr + 8 \int_o^{1/2} \rho u r \left(\frac{u^2 V^2}{2c'_{p_o} T'_o} + z \frac{g_z D}{c'_{p_o} T'_o}\right) dr\right\} \tag{113}$$

It is noted in the last term that the groups $V^2/2c'_{p_o} T'_o$ and $g_z D/c'_{p_o} T'_o$ are very small; thus, the last integral is negligible. Also,

$$\Delta\left[\int_0^{1/2} H\rho urdr\right] = \int_0^{1/2} H\rho urdr\Big|_{z'} - \int_0^{1/2} H\rho urdr\Big|_0 \qquad (114)$$

$$= \int_0^{1/2} H\rho urdr\Big|_{z'}$$

It is to be noted that

$$H = \int_1^{\overline{T}} c_p d\overline{T} \qquad (115)$$

The final result is

$$\int_1^{\overline{T}_m} c_p d\overline{T} = 8 \int_0^{1/2} H\rho urdr \qquad (116)$$

which is solved for $\overline{T}_m$. The integral on the right was evaluated by Simpson's rule. If c_p' is constant, then

$$\overline{T}_m = 1 + 8\int_0^{1/2} H\rho urdr \qquad (117)$$

If $c_p' = c_o + c_1 T'$, then

$$\overline{T}_m = \frac{-c_o + \sqrt{c_o^2 + 2c_1 T_o'\left(c_o + \frac{c_1 T_o'}{2} + 8\rho_o \int_o^{1/2} H\rho urdr\right)}}{c_1 T_o'} \qquad (118)$$

The temperature T_m defined by equation (110) is actually a generalization of the cup-mixing temperature.

Local Nusselt Number.--To find an expression for the numerical evaluation of the local Nusselt number, we first note that by definition

$$h = \frac{k'_w \left.\frac{\partial T'}{\partial r'}\right|_{r'=R}}{(T'_w - T'_m)} \tag{119}$$

Then, putting in non-dimensional form,

$$Nu = \frac{hD}{k'_m} = \frac{k'_w \left.\frac{\partial T}{\partial r}\right|_{r=1/2}}{k'_m(1 - T_m)} \tag{120}$$

(Note that we choose to define Nu such that k' is evaluated at the mean temperature.) Using equation (75) to approximate $\left.\frac{\partial T'}{\partial r'}\right|_{r=1/2}$, the numerical approximation becomes

$$Nu = \frac{k'_w(3T_{N+1} - 4T_N + T_{N-1})}{k'_m(1 - T_m)(2\Delta r)} \tag{121}$$

Relative Friction Factor.--For design purposes, perhaps the most convenient way to present the pressure drop data calculated in this study is in the form of "relative friction factors." It will be recalled that

$$G = \frac{\frac{Re}{Fr}\left[\int_o^{1/2} \rho r\,(P3)dr\right] - 1/8}{\frac{Re}{2}\int_o^{1/2} \rho r\,(P2)dr} \tag{122}$$

Thus G, the pressure gradient, may be broken up into two parts. The first term, $\frac{2\int_o^{1/2} \rho r(P3)dr}{Fr\int_o^{1/2} \rho r(P2)dr}$ is due to the gravity head; the second term, $\frac{1}{-4Re\int_o^{1/2} \rho r(P2)dr}$, is the pressure drop caused by fluid friction.

Rewritten in simpler form, equation (122) is

$$G = G_g + G_f \tag{123}$$

Examination of the results of this study shows that G_g may be approximated closely by

$$G_g = \rho_m/Fr \tag{124}$$

Then

$$G = \rho_m/Fr + G_f \tag{125}$$

To obtain the total pressure drop, the equation is integrated to give

$$\Delta p = \frac{1}{Fr} \int_o^z \rho_m dz + \int_o^z G_f dz \tag{126}$$

The first integration can be performed using the data given in this study for T_m as a function of z. The second integration was performed in this study using the trapezoidal rule.

To simplify reporting the results, the concept of "relative friction factor" is introduced. The pressure drop computed is compared to the pressure drop that would have resulted had the fluid been flowing isothermally at the wall temperature. Thus,

$$\frac{\Delta p}{(\Delta p)_w} = \frac{f}{f_w} = \frac{-\int_o^z G_f dz}{(32z\mu_w/Re\rho_w)} \tag{127}$$

APPENDIX C

IMPLEMENTATION OF NUMERICAL SCHEME

The problem described in this study was solved by implementing the numerical scheme (described in detail in Appendix B) on the Burroughs 220 Data Processing System at the Rich Electronic Computer Center of the Georgia Institute of Technology. The program employed the Burroughs Algebraic Compiler, which is a representation of Algol for use with the Burroughs 220. A description of this language is found in reference (31). Card equipment with Fortran characters was used.

For completeness, the program used is presented at the conclusion of this Appendix. Since the programming language used is not universally applicable, and since the language will doubtless change with passing time, it is explained below what the program does and how it does it, so that the ideas presented here may be implemented on another machine if desired.

It should be pointed out that the dimensionless temperature $\overline{T}$, rather than T, was used for the actual computations in the program. $\overline{T}$ is much easier to handle in many situations in the program. All results are printed out as T, rather than $\overline{T}$.

Physical property data for the various substances were fitted by empirical equations. In some cases, the particular form of the equation depended on the substance being investigated. A complete discussion of the equations used is given in Appendix E.

Perhaps the clearest way to discuss the program is by reference

to a flowsheet. A simplified flowsheet is shown in Figure 33, and is explained block by block below. This explanation covers only the essential features of the program, and does not mention all the details of the program, many of which are irrevelant to an understanding of the program.

1. Start.
   a. Read in Re, D, T_o', $\overline{T}_w$, constants for physical property equations, the grid spacing, and a constant telling whether the flow is upward or downward.
   b. Compute Pr, Fr, μ_o/μ_w, and Fc.
   c. Print out Re, Pr, Δr, $(\Delta z)_o$, Fr, $\overline{T}_w$, T_o', the constants in the physical property equations, D, μ_o/μ_w, and Fc.
   d. Compute the initial velocity profile (parabolic).
2. Calculate temperature profile at $z = (\Delta z)_o$.
   a. Compute $\overline{T}_{i,2}$ from equations (90).
3. Evaluate μ, ρ, c_p, k.
   a. Set $\mu_i = \rho_i = k_i = 1$ for i = 1,..., N-1.
   b. Evaluate properties at point N at temperature $\overline{T}_{N,2}$.
   c. Evaluate properties at point N+1 at $\overline{T}_w$.
   d. Print out z, Pe/z, Pe_w/z.
4. Compute pressure gradient and axial velocity profile.
   a. Evaluate P1, P2, P3 by scheme (103).
   b. Evaluate integrals $\int_o^{1/2} \rho r(P3)dr$ and $\int_o^{1/2} \rho r(P2)dr$ by Simpson's rule.

c. Evaluate G by equation (101).

d. Compute u by equation (102).

e. Compute f/f_w.

5. Calculate radial velocity at selected intervals.

a. Calculate $\bar{v}$ from equation (109).

b. Calculate T from $\bar{T}$.

c. Calculate Pe_L/z.

d. Calculate $u\rho$.

e. Print out r, u, T, $\bar{v}$, Pe_L/z, $u\rho$.

6. Compute mean temperature.

a. Compute $\bar{T}_m$ from equation (117) or (118).

b. Compute Pe_A/z and Pe_m/z.

7. Compute local Nusselt number.

a. Compute Nu from equation (121).

b. Print out T_m, Nu, G_f, G, f/f_w, Pe_A/z, Pe_m/z.

8. Test for instability of flow.

a. If $u_{N,n+1} \leq 0$, stop.

b. If $u_{1,n+1} \leq 0$, stop.

9. Test for changing Δz.

a. If $\Delta z \geq 0.003$ Pe, go on to box 10.

b. For the first 30 steps, if Nu has changed less than three per cent in a single step, double Δz.

c. For succeeding steps, if Nu changes less than five per cent in a single step, double Δz.

10. Increase z by Δz.

11. If $z \leq Pe/10$, stop.

12. Calculate temperature profile.

    a. Print out z, Pe/z, Pe_w/z.

    b. Compute temperature profile from equations (84), (86), (87) and (88).

13. Evaluate μ, ρ, c_p, k, then go to box 4.

The program was written in a rather general form, so that such quantities as the grid spacing Δr (and accordingly, the number of grid intervals N) could be varied to find a suitable size. Likewise, the scheme allows for variable $(\Delta z)_o$.

The computer program presented below is not intended to be a polished, highly efficient program. There are many improvements and omissions that can be made; this program is really an evolved product based on many other ideas which proved less successful and were discarded.

The empirical equations for viscosity and density in the program presented were those used for oil A. The form of these equations was changed when other substances were investigated.

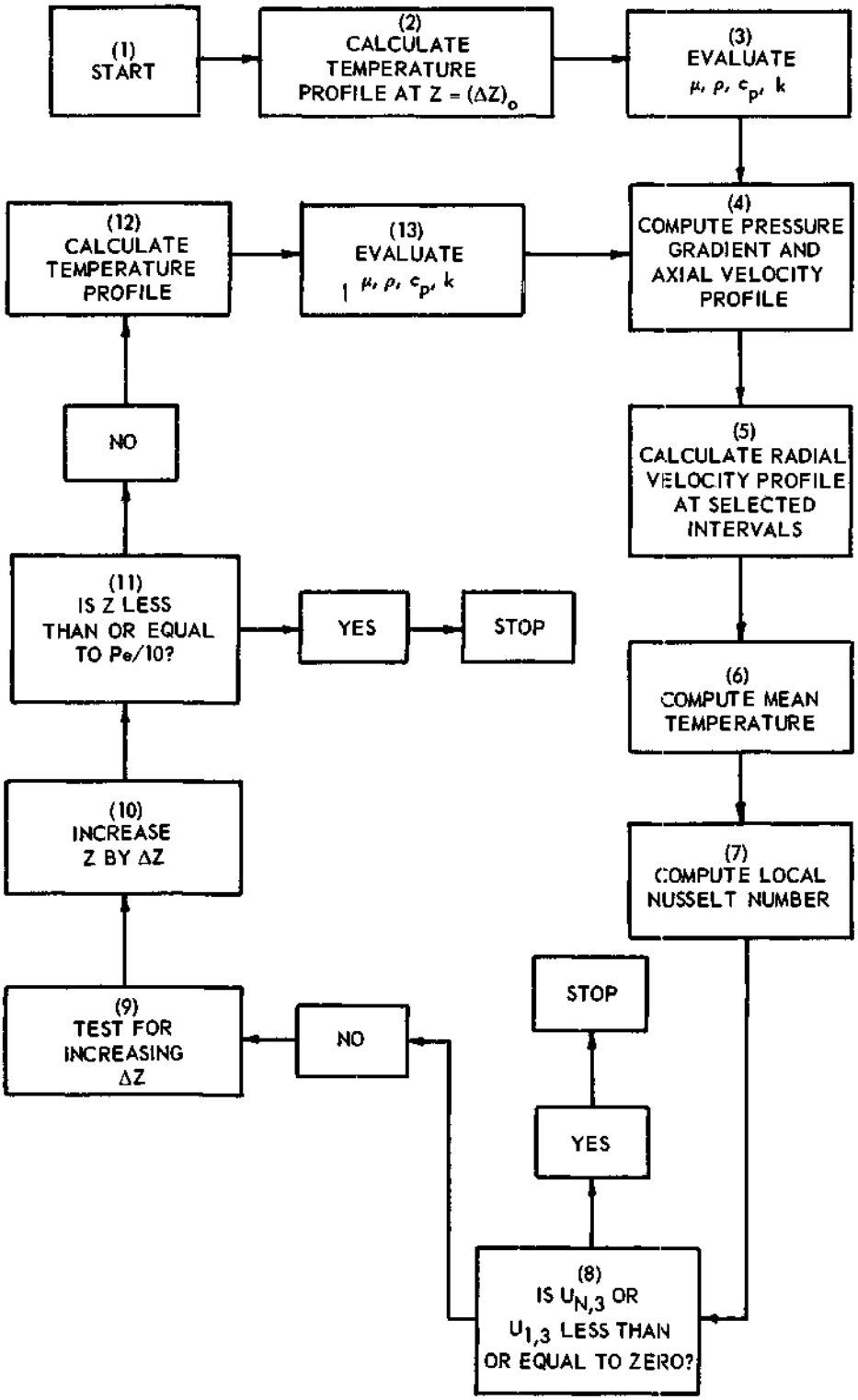

Figure 33. Flowsheet of Computer Program.

COMPUTER PROGRAM

BURROUGHS ALGEBRAIC COMPUTER - STANDARD VERSION 2/1/62

```
INTEGER I,J,L,N                                                    $
ARRAY U(51,3),GAPP(2),T(51,3),NU(2),BETA(51),GAMMA(51),
                RHO(51,3),VIS(51),CP(51,3),(51),BT(51),
        P1(51),P2(51),P3(51)                                       $
FUNCTION FCP(I,N)=(C0+C1.T(I,N)T0+C2.T(I,N)T(I,N)T0.T0)/CP0        $
FUNCTION FK(I)=(K0+K1.T(I,2)T0+K2.T(I,2)T(I,2)T0.T0)/K01           $
FUNCTION FVIS(I)=((5.8586**-3)EXP(4920.2/T0.T(I,3))÷ (8.8578**-10)
        EXP(13624.0/T0.T(I,3)))/V1                                 $
FUNCTION FRHO(I,N)=(R0+R1.T(I,N)T0+R2.T(I,N)T(I,N)T0.T0)/RH00      $
FUNCTION DT(I) = (BT(I) + 4.0)T(I,1) - T(I+1,1) - T(I-1,1)
        +X.(T(I+1,2) - T(I-1,2))   ((                    -1.0)
        /((I-1)X) - ((K(I+1) - K(I-1))          /(2K(I)X)))        $
TRANS.. READ($$DATA)                                               $
        T(1,3) = T(1,2) = RH00 = K01 = CP0 = V1 = 1.0              $
        RH00 = FRHO(1,3)                                           $
        CP0 = FCP(1,3)                                             $
        K01 = FK(1)                                                $
        V1 = FVIS(1)                                               $
        PR = V1.CP0/K01                                            $
        V2 = RE.V1/DIA.RH00(3600.0)                                $
        F = F(32.17DIA)/V2.V2                                      $
        T(N+1,3) = TW                                              $
        RATIO = 1.0/FVIS(N+1)                                      $
```

```
        SAM = 2RE.F(1.0-FRHO(N+1,3))/(1.0+FRHO(N+1,3))                    $
        WRITE($$NO1,FMT1)                                                  $
        FOR I = (1,1,N+1)                                                  $
BEGIN           U(I,2) =          2.0(1.0 - 4(I-1)(I-1)X.X)                $
        T(I,3)=T(I,2)=RHO(I,2)=K(I)=CP(I,2)=VIS(I)=RHO(I,3)=
        CP(I,3) = 1.0                                    END               $
        COW = 1.0                                                          $
        NU(1) = RATIO = 0.0                                                $
        T(N+1,2) = (1.0 + TW)/2.0                                          $
        T(N,3) = 1.0 + DEL(TW-1.0)(0.5/(N-1)+1.0)/2X.X.RE.PR.U(N,2)        $
        T(N+1,3) = TW                                                      $
        TCUM = DEL                                                         $
        TTOT = RE.PR/10                                                    $
        TTEST = DEL                                                        $
        FOR I = N,N+1                                                      $
BEGIN   RHO(I,3) =              FRHO(I,3)                                  $
        CP(I,3) =           FCP(I,3)                                       $
        K(I) = FK(I)                                                       $
        VIS(I) = FVIS(I)                                      END          $
        SAM = -32.0VIS(N+1)/RHO(N+1,3)RE                                   $
        BUG = RE.PR/DEL                                                    $
        DP2 = BUG.CP(N+1,3)/K(N+1)                                         $
        WRITE($$NO2,FMT2)                                                  $
        GO TO AXV                                                          $
FROG..  BUG = RE.PR/TCUM                                                   $
        DP2 = BUG.CP(N+1,3)/K(N+1)                                         $
```

```
        WRITE($$NO2,FMT2)                                                    $
COMMENT CALCULATE TEMPERATURE PROFILE                                        $
        FOR I = (2,1,N)                                                      $
        BT(I) = -((X.X.RE.PR.RHO(I,2)CP(I,2)U(I,2))/(K(I)DEL))-2.0           $
        BETA(2) = 1.5BT(2) + 2.0                                             $
        GAMMA(2) = 1.5DT(2)/BETA(2)                                          $
        FOR I = (3,1,N)                                                      $
BEGIN   BETA(I) = BT(I) - (1.0/BETA(I-1))                                    $
        GAMMA(I) = (DT(I) - GAMMA(I-1))/BETA(I)          END                 $
        T(N+1,3) = TW                                                        $
        FOR I = (N,-1,2)                                                     $
        T(I,3) = GAMMA(I) - (T(I+1,3)/BETA(I))                               $
        T(1,3) = (4T(2,3) - T(3,3)) / 3                                      $
        FOR I = (1,1,N+1)                                                    $
BEGIN   RHO(I,3) = FRHO(I,3)                                                 $
        VIS(I) = FVIS(I)                                                     $
        CP(I,3) = FCP(I,3)                               END                 $
COMMENT CALCULATE AXIAL VELOCITY PROFILE                                     $
AXV..   P1(1) = P2(N+1) = P3(N+1) = 0.0                                      $
        P1(2) = X.X(14  RHO(2,3)-2RHO(3,3))/24.0                             $
        FOR I = (3,1,N+1)                                                    $
        P1(I)=P1(I-2)+X.X((I-1)RHO(I,3)+4.0(I-2)RHO(I-1,3)
        + (I-1)RHO(I-2,3))/3                                                 $
        P2(N) = X.X(-19.0(N-4)/VIS(N-3) + 106.0(N-3)/VIS(N-2)
        -264.0(N-2)/VIS(N-1) + 646.0(N-1)/VIS(N) + 251.0(N )
        /VIS(N+1))/720.0                                                     $
```

```
        P3(N) = (-19.0P1(N-3)/(N-4)VIS(N-3) + 106.0P1(N-2)/(N-3)
        VIS(N-2) - 264.0P1(N-1)/(N-2)VIS(N-1) + 646.0P1(N)/(N-1)
        VIS(N) + 251.0P1(N+1)/N.VIS(N+1))/720.0                          $
        P2(N-1) = P2(N) + X.X((N-4)/VIS(N-3) - 5.0(N-3)/VIS(N-2)
        + 19.0(N-2)/VIS(N-1) + 9.0(N-1)/VIS(N))/24.0                     $
        P3(N-1) = P3(N) + (P1(N-2)/(N-4)VIS(N-3) - 5.0P1(N-2)/(N-3)
        VIS(N-2) + 19.0P1(N-1)/(N-2)VIS(N-1)+9.0P1(N)/(N-1)VIS
        (N))/24                                                          $
        FOR I = (N-2,-1,2)                                               $
BEGIN   P2(I) = P2(I+2) + X.X((I+1)/VIS(I+2) + 4.0(I)/VIS(I+1)
        + (I-1)/VIS(I))/3.0                                              $
        P3(I) = P3(I+2) + (P1(I+2)/(I+1)VIS(I+2) + 4.0P1(I+1)/I.
        VIS(I+1) + P1(I)/(I-1)VIS(I))/3.0              END               $
        DP1 = DP2 = DP3 = DP4 = 0.0                                      $
        FOR I = (2,2,N)                                                  $
BEGIN   DP2 = DP2 + RHO(I,3)(I-1)P3(I)                                   $
        DP1 = DP1 + RHO(I+1,3)I.P3(I+1)                                  $
        DP4 = DP4 + RHO(I,3)(I-1)P2(I)                                   $
        DP3 = DP3 + RHO(I+1,3)I.P2(I+1)                END               $
        INT1 = 0.66666667(2DP2 + DP1)X.X                                 $
        INT2 = 0.66666667(2DP4 + DP3)X.X                                 $
        GFRIC = -1/4RE.INT2                                              $
        G = GFRIC + 2F.INT1/INT2                                         $
        FOR I = (2,1,N)                                                  $
        U(I,3) = RE(F.P3(I) - G.P2(I)/2)                                 $
        U(N+1,3) = 0.0                                                   $
```

```
        U(I,3) = (4U(2,3) - U(3,3))/3                                    $
        GAPP(2) = GFRIC/SAM                                              $
        IF COW EQL 1.0                                                   $
        GAPP(1) = GAPP(2)                                                $
        RATIO = ((TCUM - DEL)RATIO + 0.5DEL(GAPP(2) + GAPP(1)))
        /TCUM                                                            $
        IF (TCUM GEQ TTEST) OR (U(N,3) LEQ 0.0 ) OR (U(1,3) LEQ 0.0)     $
BEGIN   P1(N+1) = 0.0                                                    $
        L = N/10                                                         $
        FOR I = (N+1-L,-L,1)                                             $
        P1(I) = P1(I+L) + L.X. X ((I+L-1)(RHO(I+L,3)U(I+L,3)
        -RHO(I+L,2)U(I+L,2))+(I-1)(RHO(I,3)U(I,3)-RHO(I,2)U(I,2)))
        /2DEL                                                            $
        WRITE ($$FMT3A)                                                  $
        FOR I = (1,L,N+1)                                                $
BEGIN   W = (I-1)X                                                       $
        T(21+I,3) = (T(I,3) - 1.0)/(TW - 1.0)                            $
        T(21+I,2) =  T(I,3)                                              $
        DP6 = BUG.CP(I,3)/FK(21+I)                                       $
        INT1 = RHO(I,3)U(I,3)                                            $
        WRITE ($$NO3,FMT3B)                          END                 $
        TTEST = TTEST + 8DEL                         END                 $
COMMENT CALCULATE MEAN TEMPERATURE                                       $
        DP1 = DP2 = 0.0                                                  $
        FOR I = (2,1,N)                                                  $
BEGIN   P1(I) = RHO(I,3)U(I,3)(I-1)                                      $
```

```
        P2(I) = (T(I,3)-1.0)(C0+0.5C1(T(I,3)+1.0)T0)/CP0                    $
        P3(I) = P1(I)P2(I)                                  END             $
        FOR I = (2,2,N)                                                     $
BEGIN   DP2 = DP2 + P3(I)                                                   $
        DP1 = DP1 + P3(I+1)                                 END             $
        HM = X.X(4DP2 + 2DP1)/3                                             $
        TM = (-C0+SQRT(C0.C0+2C1.T0(C0+0.5C1.T0+8CPC.HM)))/C1.T0            $
        T(51,3) = T(51,2) = 0.5(TM + 1)                                     $
        CP(51,3) = FCP(51,3)                                                $
        K(51) = FK(51)                                                      $
        T(50,3) = T(50,2) = TM                                              $
        DP6 = BUG.FCP(50,3)/FK(50)                                          $
        NU(2) = (3TW-4T(N,3)+T(N-1,3))K(N+1)/2X(TW-TM)FK(50)                $
        NUL = 4BUG.HM/(2TW-TM-1)K(51)                                       $
        BUG = BUG.CP(51,3)/K(51)                                            $
        TM = (TM - 1.0)/(TW - 1.0)                                          $
        WRITE ($$NO4,FMT4)                                                  $
COMMENT TEST FOR INSTABILITY OF FLOW                                        $
        IF (U(N,3) LEQ 0.0 ) OR (U(1,3) LEQ 0.0 )                           $
        STOP                                                                $
COMMENT TEST TO SEE IF DELTA L/D CAN BE INCREASED                           $
        IF DEL GEQ 0.003RE.PR                                               $
        GO TO MAXDL                                                         $
EITHER  IF COW LEQ 30.0                                                     $
        W=0.03                                                              $
OTHERWISE                                                                   $
```

```
        W = 0.05                                                    $
EITHER IF ABS((NU(1) - NU(2))/NU(2)) LEQ W                          $
BEGIN   W = 1.0                                                     $
        DEL = 2DEL                                      END         $
OTHERWISE                                                           $
MAXDL.. W = 0.0                                                     $
        TCUM = TCUM + DEL                                           $
        FOR I = (1,1,N+1)                                           $
BEGIN   IF W NEQ 0.0                                                $
        GO TO GRASS                                                 $
        T(I,1) = T(1,I)                                             $
GRASS.. U(I,2) = U(I,3)                                             $
        T(I,2) = T(I,3)                                             $
        RHO(I,2) = RHO(I,3)                                         $
        CP(I,2) = CP(I,3)                                           $
        K(I) = FK(I)                                    END         $
        GAPP(1) = GAPP(2)                                           $
        NU(1) = NU(2)                                               $
        COW = COW + 1.0                                             $
        IF TCUM LSS TTOT                                            $
        GO TO FROG                                                  $
        STOP                                                        $
INPUT DATA(RE,F,TW,TO,DIA,X,DEL,N,RO,R1,R2,CO,C1,C2,KO,K1,K2)       $
OUTPUT NO1(RE,PR,X,DEL,F,TW,TO,RO,R1,R2,RHOO,V1,V2,CO,C1,C2,CPO,
KO,K1,K2,KO1,DIA,RATIO,SAM)                                         $
OUTPUT NO2(TCUM,BUG,DP2)                                            $
```

```
OUTPUT NO3(W,U(I,3),P1(I),T(21+I,3)DP6,INT1)                                $
OUTPUT NO4(TM,NU(2),NUL,GFRIC,G,GAPP(2),RATIO,BUG,DP6)                      $
FORMAT FMT1(*RE=*,F10.4,B5,*PR=*,F10.4,B5,*DELTA R=*,F10.4,B5,
*DELTA L/D=*,F10.4,B5,*F=*,F10.4,B5,W3,(*TW=*,F11.5,B5,*TO=*,F11.5,
B5,*RO=*,F14.8,B5,*R1=*,F14.8,B5,*R2=*,F14.8,WO,(*RHOO=*,F14.8,B5,
*VI=*,F14.8,B5,*VE=*,F14.8,B5,*CO=*,F14.8,WO,(*C1=*,F14.8,B5,*C2=*,
F14.8,B5,*CPO=*,F14.8,B5,*KO=*,F14.8,B5,*K1=*,F14.8,WO,(*K2=*,F14.8,
*KO1=*,F14.8,B5,*DIA=*,F14.8,B5,*VISR=*,F14.8,B5,*FCP=*,F14.8,W2)))))  $
FORMAT FMT2(*L/D=*,F14.8,B5,*RE.PR(D/L)=*,F14.8,B5,*MOW=*,F14.8,W4)         $
FORMAT FMT3A(B9,*R*,B16,*U*,B16,*V*,B16,*T*,B15,*MOF*,B14,*U.RHO*,W2)        $
FORMAT FMT3B(6F17.8,WO)                                                     $
FORMAT FMT4(*TMEAN=*,F14.8,B5,*NU=*,F14.8,B5,*NUMEAN=*,F14.8,B5,
*GFRIC=*,F14.8,B5,*G=*,F14.8,WO,(*GRAD=*,F14.8,B5,*RATIO=*,
F14.8,B5,*MOA=*,F14.8,B5,*MOM=*,F14.8,W2))                                  $
FINISH                                                                      $
```

APPENDIX D

STABILITY AND CONVERGENCE OF NUMERICAL SCHEME

The questions of stability and convergence of the numerical scheme used in solving the energy equation are difficult ones. Only heuristic arguments can be made here--but these arguments are based on those used by professional mathematicians (32) in cases where the coefficients of a differential equation are variable and where the boundary conditions are not of the proper type for an easy proof--or for any proof at all.

The argument used is this: an equation of the same form as the energy equation, but with constant coefficients, will be investigated. The von Neumann stability criterion is applied to this equation. In essence, this criterion consists of attempting to find a solution of the form $A\xi^{n}e^{imj\Delta x}$ to a difference equation. If such a solution is found and if $|\xi| \leq 1 + 0(\Delta t)$, then, under certain restrictions (33), the equation is said to be stable. If it is stable, it is assumed to be convergent; this assumption can be proved for large classes of linear equations. (Richtmyer (33) gives a very thorough treatment of these questions. A much more elementary, but also much less complete treatment is given by Hildebrand (34).)

If the difference scheme is found to be unconditionally stable when the coefficients are constant, it will likewise be assumed to be unconditionally stable when the coefficients are variable. This is the essence of the argument used by Richtmyer (32)--an argument with experimental and intuitive (but no theoretical) justification.

We consider the differential equation with constant coefficients

$$\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial u}{\partial x} \ , \ a > 0 \tag{128}$$

We make the finite difference approximation

$$\frac{u_{j,n+1} - u_{j,n-1}}{2\Delta t} = \frac{a}{2}\left[\frac{u_{j+1,n+1} - 2u_{j,n+1} + u_{j-1,n+1}}{(\Delta x)^2}\right. \tag{129}$$

$$\left. + \frac{u_{j+1,n-1} - 2u_{j,n-1} + u_{j-1,n-1}}{(\Delta x)^2}\right] + \frac{b}{2\Delta x}\left[u_{j+1,n} - u_{j-1,n}\right]$$

We attempt to find a solution of the form

$$u_{jn} = A\xi^n e^{imj\Delta x} \tag{130}$$

where $i = \sqrt{-1}$. After substituting and simplifying, we obtain

$$(\xi^2 - 1) + \frac{4a\Delta t}{(\Delta x)^2} \sin^2 \left(\frac{m\Delta x}{2}\right)(\xi^2 + 1) - \frac{2b(\Delta t)}{\Delta x} \xi i \sin(m\Delta x) = 0 \tag{131}$$

We let $\alpha = \frac{4a\Delta t}{(\Delta x)^2}$ and $\beta = \frac{m\Delta x}{2}$; after making these substitutions and solving for ξ, we obtain

$$\xi = \frac{i(2b\sqrt{\frac{\alpha\Delta t}{a}} \sin 2\beta)}{1 + \alpha \sin^2\beta} \pm \sqrt{-\left(\frac{2b\sqrt{\frac{\alpha\Delta t}{a}} \sin 2\beta}{1 + \alpha \sin^2\beta}\right)^2 + \left(\frac{1 - \alpha \sin^2\beta}{1 + \alpha \sin^2\beta}\right)} \tag{132}$$

Thus,

$$|\xi| \leq \left|\frac{4b\sqrt{\frac{\alpha\Delta t}{a}} \sin 2\beta}{1 + \sin^2\beta}\right| + \left|\sqrt{\frac{1 - \alpha \sin^2\beta}{1 + \alpha \sin^2\beta}}\right| \leq 1 + O(\Delta t) \tag{133}$$

Thus, the von Neumann stability criterion is satisfied for any b and any $a > 0$, for this difference equation which has the same form as the energy equation. Under the assumptions of the argument, then, the energy equation is stable and convergent.

This argument is perhaps unsatisfactory; more satisfying is the excellent agreement between the numerical scheme and the Graetz Solution for the case of constant physical properties, shown in Table 1. At least for this special case, the numerical scheme is apparently stable, convergent, and equally important, accurate.

For the explicit finite difference equation, it is likewise true that no rigorous stability and convergence arguments can be made. However, it is felt that the present explicit scheme behaves like the explicit scheme for the equation

$$\frac{\partial u}{\partial t} = a(x,t)\frac{\partial^2 u}{\partial x^2} + 2b(x,t)\frac{\partial u}{\partial x} - c(x,t)u + d(x,t) \qquad (134)$$

with slightly different boundary conditions, for which a sufficient condition for stability and convergence can be shown to be (35)

$$\Delta t \leq \frac{(\Delta x)^2}{2a(x,t) + (\Delta x)^2 c(x,t)} \qquad (135)$$

This supposition is based on the results of Richtmyer (32). In the present case, the stability condition becomes

$$\Delta z \leq \frac{(\Delta r)^2 \, u_{in} \, Pe}{2} \qquad (136)$$

This requirement was always met in this study.

APPENDIX E

PHYSICAL PROPERTY DATA

Four substances were considered in the calculations reported in this study: oil A, water, air, and helium. To investigate the heat transfer and fluid friction characteristics of the substances, it was necessary to fit their physical properties with empirical equations. These equations were fitted to the data to an accuracy usually well within one per cent. The physical property data and the empirical equations used are given in Tables 7-10.

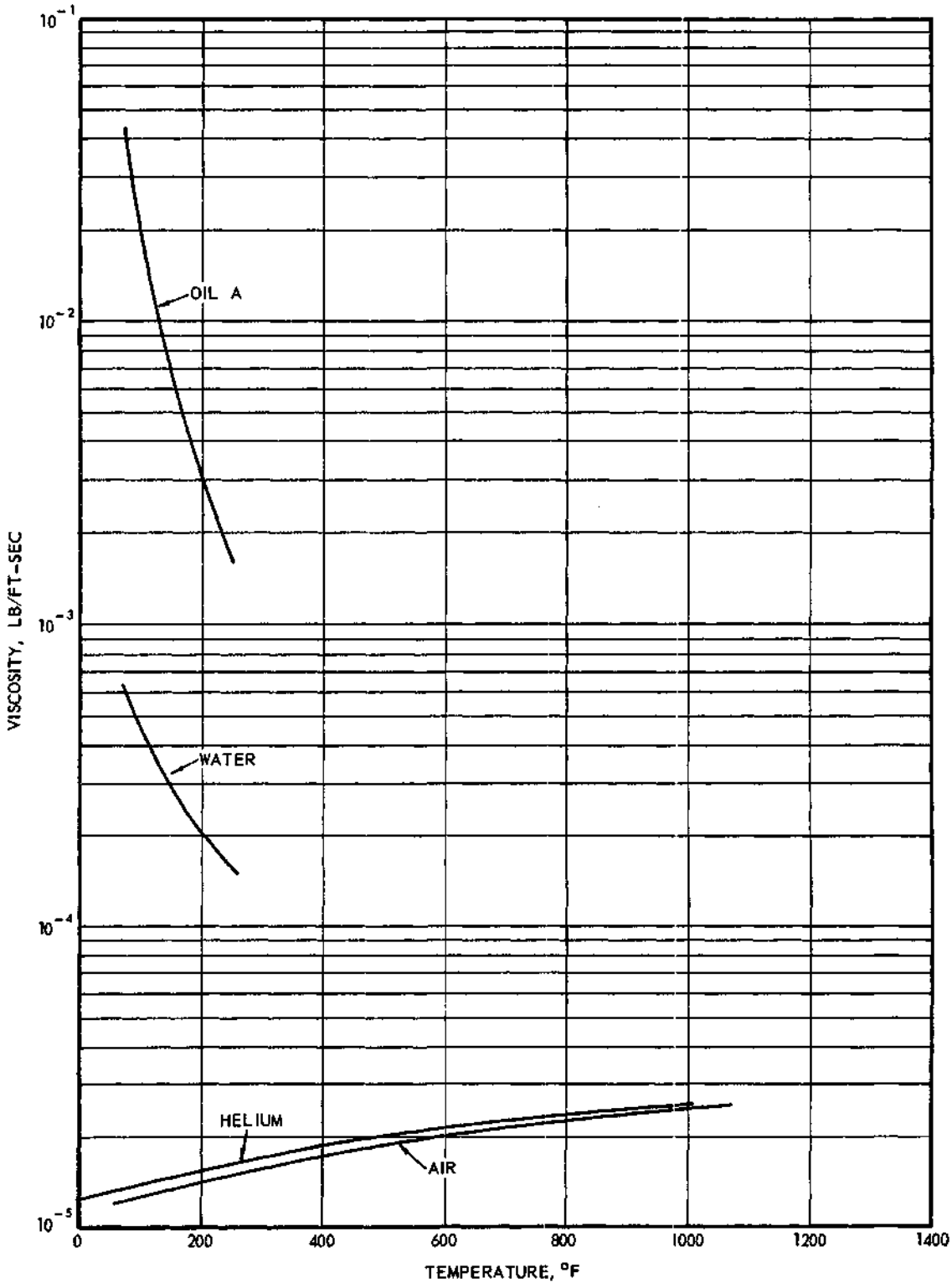

Figure 34. Viscosities of Substances Studied.

Table 7. Physical Properties of Oil A* (9).

T', °F	ρ', lb/ft^3	c_p', Btu/lb-°R	k', Btu/ft-hr-°R	μ', lb/ft-sec
90	55.47	0.450	0.07393	2.685×10^{-2}
100	55.25	0.455	0.07370	2.022×10^{-2}
120	54.81	0.466	0.07327	1.195×10^{-2}
140	54.37	0.477	0.07282	0.7829×10^{-2}
160	53.92	0.487	0.07230	0.5500×10^{-2}
180	53.48	0.498	0.07195	0.4022×10^{-2}
200	53.03	0.508	0.07150	0.3023×10^{-2}
220	52.58	0.519	0.07108	0.2377×10^{-2}
240	52.13	0.530	0.07065	0.1913×10^{-2}

Empirical equations:

$$\rho' = 67.70978 - 2.22667 \times 10^{-2}\ T' \quad (\rho' \text{ in lb/ft}^3; T', °R)$$

$$c_p' = 0.15780 + 5.31579 \times 10^{-4}\ T' \quad (c_p' \text{ in Btu/lb-°R; } T', °R)$$

$$k' = 0.085994 - 2.19474 \times 10^{-5}\ T' \quad (k' \text{ in Btu/ft-hr-°R; } T', °R)$$

$$\mu' = 1.6274 \times 10^{-6} \exp(4.9202 \times 10^{3}/T') + 2.4605 \times 10^{-13} \exp(1.3624 \times 10^{4}/T') \quad (\mu' \text{ in lb/ft-sec; } T', °R)$$

*This oil was furnished to Martinelli and co-workers by the Standard Oil Company of California for their tests.

Table 8. Physical Properties of Water (36,37).

T', °F	ρ', lb/ft^3	k', Btu/ft-hr-°R
80	62.22	0.352
100	62.00	0.362
120	61.71	0.371
140	61.38	0.378
160	61.00	0.384
180	60.58	0.388
200	60.13	0.392
220	59.63	0.394

Empirical equations:

$$\rho' = 46.85311 + 6.55268 \times 10^{-2}\, T' - 6.875 \times 10^{-5}\, T'^2$$

(ρ' in lb/ft^3; T', °R)

$$c_p' = 1.000 \quad (c_p' \text{ in Btu/lb-°R})$$

$$k' = -0.49958 + 2.58911 \times 10^{-3}\, T' - 1.875 \times 10^{-6}\, T'^2$$

(k' in Btu/ft-hr-°R; T', °R)

$$\frac{1}{\mu'} = 2.1482 \left[(T' - 8.435) + \sqrt{8078.4 + (T' - 8.435)^2} \right] - 120$$

(μ' in poises; T', °C) (37)

Table 9. Physical Properties of Air (38).

T', °F	ρ', *lb/ft^3	c_p', Btu/lb-°R	k', Btu/ft-hr-°R	μ', lb/ft-sec
80	0.0735	0.2402	0.01516	1.241×10^{-5}
170	0.0623	0.2410	0.01735	1.394×10^{-5}
260	0.0551	0.2422	0.01944	1.536×10^{-5}
350	0.0489	0.2438	0.02142	1.669×10^{-5}
440	0.0440	0.2459	0.02333	1.795×10^{-5}
530	0.0401	0.2482	0.02519	1.914×10^{-5}
620	0.0367	0.2520	0.02692	2.028×10^{-5}
710	0.0339	0.2540	0.02862	2.135×10^{-5}
800	0.0314	0.2568	0.03022	2.239×10^{-5}
890	0.0294	0.2593	0.03183	2.339×10^{-5}
980	0.0275	0.2622	0.03339	2.436×10^{-5}
1070	0.0259	0.2650	0.03483	2.530×10^{-5}

Empirical equations:

$$\rho' = 39.667215/T' \quad (\rho' \text{ in lb/ft}^3;\ T',\ ^\circ F)$$

$$c_p' = 0.22369358 + 2.6278162 \times 10^{-5}\ T' \quad (c_p' \text{ in Btu/lb-°R};\ T',\ ^\circ R)$$

$$k' = 1.002167 \times 10^{-3} + 2.8649806 \times 10^{-5}\ T' - 4.2876644 \times 10^{-9}\ T'^2 \quad (k' \text{ in Btu/ft-hr-°R};\ T',\ ^\circ R)$$

$$\mu' = \frac{2.63142 \times 10^{-3}\ (T')^{1.5}}{(T' + 198.74)} \quad (\mu' \text{ in lb/ft-hr};\ T',\ ^\circ R)$$

*At one atmosphere pressure.

Table 10. Physical Properties of Helium (39).

T', °F	ρ', *lb/ft^3	c_p', Btu/lb-°R	k', Btu/ft-hr-°R	μ', lb/ft-sec
0	0.0119	1.242	0.0784	1.221×10^{-5}
200	0.00829	1.242	0.0977	1.549×10^{-5}
400	0.00637	1.242	0.114	1.848×10^{-5}
600	0.00517	1.242	0.130	2.092×10^{-5}
800	0.00439	1.242	0.145	2.335×10^{-5}
1000	0.00376	1.242	0.159	2.565×10^{-5}

Empirical equations:

$$\rho' = 5.468830/T' \quad (\rho' \text{ in lb/ft}^3;\ T', °R)$$

$$c_p' = 1.242 \quad (c_p' \text{ in Btu/lb-°R})$$

$$k' = 0.03383724 + 1.0399225 \times 10^{-4}\, T' - 1.25 \times 10^{-8} T'^2$$

(k' in Btu/ft-hr-°R; T', °R)

$$\mu' = \frac{4.3956 \times 10^{-2}\, (T')^{1.5}}{(T' + 131.25)} \quad (\mu' \text{ in lb/ft-hr};\ T', °R)$$

*At one atmosphere pressure.

APPENDIX F

TABULATED RESULTS

Table 11. Mean Temperature and Local Nusselt Number for Liquids at $\mu_o/\mu_w = 1$.

Fc	Pe_m/z	T_m	Nu	Fc	Pe_m/z	T_m	Nu
± 0.1	9090	0.0137	26.7	100	4780	0.0141	29.1
	3570	0.0257	17.0		9780	0.0268	18.7
	980	0.0600	10.3		1140	0.0616	11.6
	382	0.109	7.39		368	0.129	8.01
	97.1	0.254	4.82		112	0.267	5.69
	35.4	0.452	3.93		38.2	0.476	4.52
	10.0	0.811	3.64		10.7	0.808	3.98
200	9970	0.0147	30.7	400	9890	0.0159	32.7
	3890	0.0291	19.7		3830	0.0332	21.4
	1050	0.0724	12.0		1070	0.0828	13.4
	384	0.141	8.75		384	0.167	9.91
	127	0.280	6.56		110	0.360	7.68
	37.3	0.526	5.15		39.1	0.573	6.30
	10.7	0.833	4.00		26.1	0.664	5.81
800	9770	0.0180	35.6	-100	9520	0.0123	25.9
	3750	0.0396	24.1		3960	0.0210	16.6
	1180	0.0935	15.8		1127	0.0454	9.92
	536	0.163	12.8		362	0.0908	6.55
	215	0.303	10.7		120	0.181	4.59
					38.9	0.371	3.47
					10.6	0.750	3.39
-200	9720	0.0108	24.5				
	3500	0.0186	14.4				
	1190	0.0333	9.21				
	400	0.0616	6.01				
	211	0.0925	4.78				

Table 12. Mean Temperature and Local Nusselt Number for Liquids at $\mu_o/\mu_w = 2.$

Fc	Pe_m/z	T_m	Nu	Fc	Pe_m/z	T_m	Nu
± 0.1	9220	0.0157	31.3	100	9220	0.0166	32.6
	3360	0.0315	19.2		3210	0.0360	20.1
	1060	0.0650	11.8		1070	0.0746	12.9
	322	0.134	7.63		324	0.160	8.56
	115	0.244	5.44		110	0.303	6.32
	32.5	0.488	4.05		34.0	0.548	4.93
	10.6	0.805	3.77		10.4	0.848	4.05
200	9340	0.0172	34.0	400	8700	0.0193	35.2
	3400	0.0375	21.8		3480	0.0409	24.2
	1070	0.0833	13.8		1030	0.0998	15.2
	351	0.173	9.63		308	0.226	10.8
	88.7	0.396	6.84		124	0.396	9.13
	35.7	0.591	5.89				
	10.6	0.871	4.34				
800	9230	0.0195	39.2	-100	9120	0.0147	29.6
	3360	0.0479	27.3		3320	0.0275	17.6
	1000	0.122	17.7		1040	0.0528	10.6
	318	0.275	13.4		308	0.104	6.52
					94.0	0.208	4.31
					31.9	0.410	3.34
					10.4	0.759	3.40
-200	9220	0.0132	27.9				
	3360	0.0220	15.9				
	1000	0.0367	9.03				
	568	0.0459	7.06				

Table 13. Mean Temperature and Local Nusselt Number for Liquids at $\mu_o/\mu_w = 5$.

Fc	Pe_m/z	T_m	Nu	Fc	Pe_m/z	T_m	Nu
± 0.1	8780	0.0181	36.5	100	8780	0.0190	37.9
	3080	0.0390	22.6		3080	0.0431	24.4
	967	0.0787	13.2		1000	0.0915	14.9
	312	0.150	8.40		316	0.189	9.85
	97.5	0.284	5.57		104	0.364	7.30
	31.5	0.513	4.23		34.5	0.613	5.96
	11.3	0.798	3.79		11.2	0.871	4.51
200	8040	0.0212	38.0	400	8680	0.0206	41.0
	3520	0.0418	27.5		3110	0.0505	28.7
	1050	0.0993	16.5		996	0.120	18.4
	316	0.220	11.1		358	0.248	13.6
	98.3	0.444	8.90		204	0.360	12.4
	43.9	0.637	8.00				
800	7950	0.0237	42.9	-100	8730	0.0171	35.0
	3320	0.0532	33.3		3220	0.0329	21.1
	960	0.148	21.9		1010	0.0583	11.7
	549	0.225	18.8		313	0.0966	6.82
					159	0.131	5.07
-200	8650	0.0159	32.3				
	3310	0.0256	18.8				
	1340	0.0308	11.1				

Table 14. Mean Temperature and Local Nusselt Number for Liquids at $\mu_o/\mu_w = 10.$

Fc	Pe_m/z	T_m	Nu	Fc	Pe_m/z	T_m	Nu
± 0.1	7860	0.0206	38.8	100	7860	0.0216	40.8
	3160	0.0413	26.3		3380	0.0424	29.4
	918	0.0873	14.2		996	0.100	16.9
	281	0.167	8.59		291	0.222	10.7
	94.4	0.299	5.78		87.1	0.457	8.24
	33.1	0.512	4.33		38.9	0.652	7.33
	11.7	0.799	3.79		22.5	0.767	6.58
200	7210	0.0229	41.0	400	8340	0.0215	44.1
	3100	0.0483	30.1		3050	0.0542	32.5
	1000	0.113	18.6		1000	0.133	21.5
	313	0.253	12.6		298	0.326	15.6
	109	0.494	10.9				
800	7830	0.0235	46.2	-100	7860	0.0193	37.0
	2960	0.0607	36.2		3150	0.0358	23.8
	806	0.186	24.4		879	0.0611	11.6
					335	0.0776	7.04
-200	7860	0.0178	34.6				
	3090	0.0277	20.2				

Table 15. Mean Temperature and Local Nusselt Number for Liquids at $\mu_o/\mu_w = 20.$

Fc	Pe_m/z	T_m	Nu	Fc	Pe_m/z	T_m	Nu
± 0.1	8000	0.0199	42.2	100	8000	0.0206	43.5
	3020	0.0442	28.8		3020	0.0484	31.1
	905	0.0926	15.4		878	0.119	17.8
	274	0.176	8.95		332	0.230	12.4
	96.1	0.304	6.01		121	0.449	10.1
	29.9	0.546	4.34		66.6	0.600	9.71
	12.4	0.790	3.86				
200	6780	0.0258	42.3	400	7790	0.0216	46.3
	2840	0.0537	32.3		3140	0.0532	36.4
	964	0.126	20.7		945	0.150	24.0
	373	0.256	15.2		405	0.298	19.5
	167	0.455	13.8				
800	7970	0.0235	48.1	-100	8000	0.0188	40.7
	3010	0.0594	39.9		3220	0.0369	26.7
	1230	0.138	31.2		932	0.0557	12.5
-200	7520	0.0187	36.9				
	3080	0.0278	22.1				

Table 16. Mean Temperature and Local Nusselt Number for Liquids at $\mu_o/\mu_w = 0.5$.

Fc	Pe_m/z	T_m	Nu	Fc	Pe_m/z	T_m	Nu
± 0.1	9610	0.0112	23.0	100	9610	0.0122	24.2
	3570	0.0218	14.7		3340	0.0254	15.1
	1170	0.0465	9.82		1150	0.0533	10.4
	340	0.106	6.49		337	0.121	6.99
	119	0.208	4.76		103	0.256	5.05
	34.8	0.437	3.70		33.2	0.481	4.02
	10.2	0.792	3.60		10.1	0.812	3.68
200	9870	0.0127	25.7	400	9420	0.0144	27.1
	3740	0.0254	16.6		3420	0.0307	17.4
	1120	0.0596	10.8		1140	0.0678	11.9
	374	0.125	7.66		357	0.151	8.45
	108	0.274	5.51		126	0.292	6.65
	38.3	0.478	4.46		34.7	0.549	5.20
	10.1	0.827	3.84		10.1	0.851	4.02
800	9760	0.0161	30.4	-100	9490	0.0102	21.6
	3700	0.0344	20.2		3530	0.0193	13.8
	1150	0.0810	13.5		1160	0.0399	9.11
	397	0.172	10.1		336	0.0897	5.94
	126	0.357	8.22		118	0.181	4.30
					34.4	0.405	3.39
					10.1	0.778	3.49
-200	9720	0.0088	20.5				
	3340	0.0166	12.7				
	1070	0.0331	8.11				
	329	0.0711	5.29				
	103	0.165	3.75				
	31.7	0.392	3.02				
	10.4	0.749	3.32				

Table 17. Mean Temperature and Local Nusselt Number for Liquids at $\mu_o/\mu_w = 0.2$.

Fc	Pe_m/z	T_m	Nu	Fc	Pe_m/z	T_m	Nu
± 0.1	9790	0.0083	19.0	100	9790	0.0091	19.9
	4060	0.0159	13.4		3960	0.0177	13.9
	1290	0.0356	8.97		1020	0.0464	8.69
	368	0.0849	5.97		367	0.0945	6.31
	106	0.198	4.15		106	0.216	4.43
	36.8	0.391	3.44		36.7	0.412	3.64
	10.2	0.769	3.38		10.2	0.779	3.47
200	9570	0.0100	20.5	400	9830	0.0110	22.5
	3920	0.0192	14.3		3600	0.0231	14.9
	1200	0.0451	9.58		1100	0.0549	10.0
	396	0.0978	6.80		350	0.122	7.12
	102	0.238	4.71		120	0.244	5.45
	36.4	0.433	3.86		35.2	0.476	4.21
	10.3	0.786	3.56		10.4	0.800	3.68
800	9800	0.0131	25.1	-100	9790	0.0075	18.1
	3330	0.0292	16.0		3400	0.0160	12.0
	1140	0.0642	11.3		1030	0.0363	7.82
	330	0.153	8.02		368	0.0743	5.61
	97.8	0.331	6.20		107	0.179	3.87
	37.6	0.513	5.21		36.9	0.370	3.25
	10.2	0.835	3.77		10.2	0.759	3.28
-200	9570	0.0066	17.0				
	3480	0.0135	11.4				
	1160	0.0279	7.65				
	363	0.0631	5.20				
	104	0.162	3.64				
	37.0	0.349	3.06				
	10.4	0.744	3.27				

Table 18. Mean Temperature and Local Nusselt Number for Liquids at $\mu_o/\mu_w = 0.1$.

Fc	Pe_m/z	T_m	Nu	Fc	Pe_m/z	T_m	Nu
± 0.1	9530	0.0070	16.9	100	9320	0.0077	17.4
	3660	0.0147	11.9		3690	0.0159	12.4
	1100	0.0343	7.89		1180	0.0361	8.43
	360	0.0762	5.51		374	0.0813	5.87
	111	0.174	3.94		106	0.193	4.13
	35.9	0.372	3.19		37.9	0.375	3.37
	11.1	0.719	3.24		10.7	0.737	3.29
200	9450	0.0083	18.1	400	9900	0.0090	19.3
	3990	0.0162	13.2		3460	0.0202	13.3
	1140	0.0402	8.64		1080	0.0475	9.11
	348	0.0927	5.99		388	0.0976	6.72
	105	0.207	4.31		107	0.228	4.77
	33.7	0.416	3.46		39.4	0.406	3.85
	10.4	0.754	3.37		10.6	0.761	3.39
800	9780	0.0109	22.2	-100	9530	0.0063	16.3
	3910	0.0218	15.4		3660	0.0133	11.5
	1200	0.0522	10.4		1160	0.0296	7.64
	352	0.125	7.35		366	0.0675	5.26
	111	0.264	5.63		111	0.161	3.74
	38.1	0.460	4.52		36.0	0.357	3.06
	10.7	0.784	3.69		11.1	0.711	3.18
-200	9730	0.0055	15.6				
	3980	0.0110	11.2				
	1060	0.0271	7.04				
	414	0.0539	5.22				
	119	0.140	3.64				
	36.7	0.340	2.95				
	10.5	0.722	3.13				

Table 19. Mean Temperature and Local Nusselt Number for Liquids at $\mu_o/\mu_w = 0.05$.

Fc	Pe_m/z	T_m	Nu	Fc	Pe_m/z	T_m	Nu
± 0.1	9250	0.0061	15.7	100	9560	0.0063	16.2
	4040	0.0120	11.6		4090	0.0127	12.0
	1240	0.0278	7.75		1240	0.0301	8.03
	365	0.0669	5.28		365	0.0722	5.48
	120	0.149	3.81		120	0.158	3.96
	38.7	0.328	2.99		38.6	0.338	3.09
	11.1	0.691	3.01		11.1	0.697	3.06
200	9060	0.0071	16.3	400	9500	0.0076	17.4
	3800	0.0143	12.1		3910	0.0155	12.8
	1140	0.0342	8.06		1160	0.0379	8.56
	402	0.0720	5.78		407	0.0797	6.18
	110	0.177	4.01		111	0.192	4.31
	35.1	0.369	3.15		35.4	0.385	3.35
	11.1	0.699	3.06		10.7	0.722	3.15
800	7980	0.0104	17.8	-100	9560	0.0054	15.4
	3580	0.0193	13.4		4090	0.0108	11.3
	1110	0.0458	9.24		1240	0.0252	7.48
	401	0.0949	6.85		366	0.0612	4.99
	110	0.223	4.88		114	0.145	3.60
	34.5	0.424	3.75		38.0	0.322	2.91
	10.6	0.743	3.25		11.0	0.688	2.94
-200	9740	0.0048	15.0				
	3700	0.0105	10.5				
	1060	0.0250	6.78				
	343	0.0581	4.75				
	111	0.139	3.45				
	35.5	0.328	2.77				
	11.2	0.677	2.90				

Table 20. Mean Temperature and Local Nusselt Number for Gases at $\mu_o/\mu_w = 1$.

Fc	Pe_m/z	T_m	Nu	Fc	Pe_m/z	T_m	Nu
± 0.1	9090	0.0137	26.7	100	8600	0.0154	25.1
	3570	0.0257	17.0		3340	0.0299	16.7
	980	0.0600	10.3		1070	0.0647	10.9
	382	0.109	7.39		305	0.147	7.29
	97.1	0.254	4.82		96.7	0.298	5.33
	35.4	0.452	3.93		33.9	0.520	4.46
	10.0	0.811	3.64		11.5	0.812	3.97
200	8200	0.0169	25.5	400	8200	0.0183	27.1
	3160	0.0336	17.2		3280	0.0368	18.9
	1050	0.0722	11.5		1060	0.0827	12.7
	321	0.159	7.99		323	0.187	9.11
	98.6	0.331	6.00		107	0.377	7.37
	34.3	0.561	5.09		35.3	0.619	6.33
	11.5	0.837	4.20		11.7	0.922	4.35
800	8600	0.0196	30.0	-100	8190	0.0137	22.4
	3220	0.0434	21.1		3050	0.0251	14.3
	965	0.107	14.2		1030	0.0486	9.29
	378	0.208	11.2		337	0.0957	6.22
	211	0.309	10.3		98.8	0.206	4.19
					31.0	0.431	3.31
					11.1	0.752	3.37
-200	8400	0.0121	21.3				
	3070	0.0206	13.2				
	945	0.0385	8.06				
	264	0.0790	4.94				

Table 21. Mean Temperature and Local Nusselt Number for Gases at $\mu_o/\mu_w = 1.5$.

Fc	Pe_m/z	T_m	Nu	Fc	Pe_m/z	T_m	Nu
± 0.1	5890	0.0195	18.2	100	6100	0.0204	19.1
	2500	0.0349	13.3		2540	0.0382	14.1
	764	0.0750	8.80		800	0.0841	9.64
	265	0.145	6.23		270	0.171	7.04
	90.0	0.278	4.62		82.8	0.357	5.53
	30.7	0.510	3.91		28.9	0.622	4.97
	13.5	0.755	3.79		14.2	0.818	4.59
200	6320	0.0209	20.0	400	6570	0.0218	21.4
	2430	0.0425	14.6		2470	0.0471	15.8
	790	0.0941	10.2		679	0.123	10.9
	272	0.195	7.77		315	0.212	9.30
	88.8	0.397	6.57		122	0.405	8.45
	30.2	0.678	-				
	22.2	0.776	-				
800	5900	0.0263	22.5	-100	6320	0.0171	17.8
	2670	0.0510	18.0		2430	0.0310	12.3
	851	0.125	13.4		755	0.0613	7.94
	423	0.213	11.7		274	0.109	5.48
	234	0.331	11.3		84.0	0.214	3.71
					27.2	0.428	2.99
					13.2	0.657	2.91
-200	6550	0.0149	17.0				
	2600	0.0243	11.6				
	795	0.0419	7.08				
	394	0.0563	5.32				

Table 22. Mean Temperature and Local Nusselt Number for Gases at $\mu_o/\mu_w = 2$.

Fc	Pe_m/z	T_m	Nu	Fc	Pe_m/z	T_m	Nu
± 0.1	4540	0.0236	13.5	100	4930	0.0239	14.5
	1860	0.0435	10.3		1770	0.0502	10.7
	622	0.0833	7.36		615	0.104	8.00
	176	0.192	5.19		216	0.210	6.20
	60.1	0.363	4.00		66.7	0.449	5.46
	20.8	0.639	3.77		27.4	0.713	-
	15.6	0.732	3.70		17.7	0.852	-
200	4550	0.0268	14.7	400	4800	0.0278	15.8
	1790	0.0540	11.3		1990	0.0560	12.6
	655	0.112	8.71		555	0.148	9.49
	223	0.239	7.04		322	0.222	8.84
	82.0	0.487	6.54		152	0.398	8.54
	50.5	0.617	-				
800	4800	0.0306	17.2	-100	4540	0.0214	12.8
	1760	0.0707	13.6		2020	0.0354	9.84
	569	0.176	11.2		556	0.0739	6.36
	278	0.316	10.9		217	0.121	4.67
					60.4	0.236	3.10
-200	4650	0.0182	12.1				
	2040	0.0277	8.99				
	597	0.0448	5.59				
	450	0.0485	4.94				

Table 23. Mean Temperature and Local Nusselt Number for Gases at $\mu_o/\mu_w = 0.667$.

Fc	Pe_m/z	T_m	Nu	Fc	Pe_m/z	T_m	Nu
± 0.1	9060	0.0125	24.5	100	9060	0.0137	25.9
	3820	0.0226	17.5		3650	0.0258	18.1
	1020	0.0546	10.7		1250	0.0536	12.2
	391	0.102	7.72		358	0.121	7.97
	118	0.216	5.15		113	0.241	5.48
	39.0	0.411	3.86		38.3	0.434	4.07
	11.5	0.746	3.35		11.4	0.756	3.46
200	9820	0.0140	28.8	400	9280	0.0164	30.7
	3610	0.0282	18.8		3840	0.0306	20.5
	1240	0.0590	12.8		1270	0.0667	14.0
	355	0.133	8.44		354	0.153	9.29
	118	0.255	5.95		116	0.290	6.69
	35.9	0.467	4.25		35.4	0.500	4.72
	11.2	0.770	3.53		11.1	0.786	3.68
800	9999	0.0179	37.9	-100	9070	0.0111	23.0
	3470	0.0389	22.2		4000	0.0191	16.9
	1290	0.0783	15.9		1300	0.0399	10.9
	351	0.185	10.7		396	0.0871	7.19
	114	0.347	8.09		119	0.193	4.79
					39.4	0.389	3.66
					11.5	0.736	3.26
-200	9850	0.0090	22.3				
	3630	0.0168	14.9				
	1190	0.0333	9.71				
	388	0.0714	6.50				
	120	0.169	4.43				
	39.7	0.367	3.47				
	11.6	0.726	3.19				

Table 24. Mean Temperature and Local Nusselt Number for Gases at $\mu_o/\mu_w = 0.5$.

Fc	Pe_m/z	T_m	Nu	Fc	Pe_m/z	T_m	Nu
± 0.1	8760	0.0116	24.5	100	9999	0.0115	26.2
	3890	0.0211	17.9		3750	0.0239	18.4
	1110	0.0494	11.4		1090	0.0552	11.9
	381	0.0997	7.73		376	0.110	8.12
	134	0.193	5.38		132	0.206	5.62
	37.6	0.408	3.70		37.4	0.417	3.82
	11.7	0.727	3.37		11.7	0.729	3.39
200	9470	0.0130	26.5	400	9720	0.0144	27.9
	3650	0.0263	19.0		4040	0.0275	21.5
	1080	0.0603	12.4		1240	0.0621	14.1
	371	0.119	8.59		386	0.130	9.47
	122	0.227	5.73		121	0.248	6.27
	36.2	0.430	3.91		36.1	0.445	4.18
	11.6	0.734	3.46		11.6	0.739	3.50
800	9970	0.0167	31.4	-100	9000	0.0101	23.8
	3870	0.0332	23.4		3950	0.0184	17.2
	1210	0.0747	15.5		1260	0.0396	11.4
	342	0.164	10.3		371	0.0918	7.08
	121	0.286	7.25		110	0.208	4.69
	39.4	0.456	4.96		35.0	0.420	3.27
	11.2	0.759	3.65		11.4	0.734	3.83
-200	9530	0.0083	22.6				
	3550	0.0163	15.7				
	1230	0.0335	10.4				
	404	0.0758	7.15				
	121	0.183	4.78				
	33.1	0.426	3.13				
	11.2	0.738	3.85				

Table 25. Velocity and Temperature Profiles for Liquids at $\mu_o/\mu_w = 1.$

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
± 0.1	5880	0.000	2.000	5880	0.000	1.680	5880	0.000	1.280	5880	0.003	0.720	5880	0.160	0.380
	3850	0.000	2.000	3850	0.000	1.680	3850	0.000	1.280	3850	0.009	0.720	3850	0.242	0.380
	2380	0.000	2.000	2380	0.000	1.680	2380	0.000	1.280	2380	0.030	0.720	2380	0.342	0.380
	1160	0.000	2.000	1160	0.000	1.680	1160	0.000	1.280	1160	0.110	0.720	1160	0.485	0.380
	435	0.000	2.000	435	0.000	1.680	435	0.015	1.280	435	0.304	0.720	435	0.642	0.380
	193	0.000	2.000	193	0.006	1.680	193	0.097	1.280	193	0.476	0.720	193	0.742	0.380
	97.1	0.001	2.000	97.1	0.048	1.680	97.1	0.240	1.280	97.1	0.611	0.720	97.1	0.812	0.380
	30.0	0.134	2.000	30.0	0.329	1.680	30.0	0.548	1.280	30.0	0.787	0.720	30.0	0.898	0.380
	13.5	0.505	2.000	13.5	0.634	1.680	13.5	0.760	1.280	13.5	0.889	0.720	13.5	0.947	0.380
100	6710	0.000	1.955	6710	0.000	1.651	6710	0.000	1.271	6710	0.002	0.738	6860	0.125	0.413
	4090	0.000	1.939	4090	0.000	1.641	4090	0.000	1.268	4090	0.007	0.746	4240	0.208	0.424
	2190	0.000	1.910	2190	0.000	1.623	2190	0.000	1.264	2200	0.031	0.761	2310	0.331	0.442
	817	0.000	1.840	817	0.000	1.581	817	0.001	1.259	839	0.143	0.799	893	0.512	0.477
	323	0.000	1.734	323	0.001	1.526	324	0.028	1.264	342	0.332	0.850	361	0.654	0.512
	94.5	0.002	1.546	95.3	0.049	1.460	98.2	0.220	1.306	104	0.581	0.914	108	0.795	0.543
	29.4	0.178	1.558	30.2	0.343	1.516	31.3	0.541	1.346	32.5	0.778	0.897	33.1	0.893	0.515
	13.9	0.540	1.788	14.2	0.650	1.627	14.5	0.765	1.348	14.8	0.889	0.835	14.9	0.947	0.463
200	6150	0.000	1.908	6150	0.000	1.620	6150	0.000	1.261	6150	0.002	0.757	6290	0.129	0.446
	3920	0.000	1.877	3920	0.000	1.601	3920	0.000	1.255	3920	0.007	0.711	4060	0.202	0.467
	1960	0.000	1.813	1960	0.000	1.561	1960	0.000	1.246	1970	0.034	0.802	2070	0.333	0.506
	743	0.000	1.673	743	0.000	1.478	743	0.002	1.234	762	0.150	0.876	811	0.508	0.572
	353	0.000	1.509	353	0.000	1.390	354	0.020	1.238	371	0.290	0.954	393	0.621	0.626
	172	0.000	1.300	173	0.009	1.293	175	0.091	1.266	186	0.440	1.035	195	0.717	0.673
	52.6	0.050	1.000	53.7	0.165	1.237	55.6	0.367	1.354	58.5	0.677	1.098	60.2	0.843	0.685
	16.4	0.507	1.472	16.6	0.613	1.492	17.0	0.733	1.367	17.4	0.872	0.936	17.6	0.938	0.543
400	4750	0.000	1.797	4750	0.000	1.549	4750	0.000	1.237	4750	0.003	0.801	4870	0.149	0.524
	3020	0.000	1.732	3020	0.000	1.507	3020	0.000	1.226	3030	0.011	0.830	3140	0.223	0.565
	1510	0.000	1.597	1510	0.000	1.423	1510	0.000	1.206	1520	0.045	0.898	1610	0.349	0.644
	615	0.000	1.327	615	0.000	1.265	616	0.003	1.187	632	0.155	1.035	672	0.505	0.764
	248	0.000	0.900	248	0.002	1.042	250	0.038	1.204	262	0.325	1.223	277	0.642	0.892
	79.2	0.031	0.266	80	0.087	0.821	82.0	0.232	1.324	86.8	0.564	1.410	90.0	0.784	0.977
	24.4	0.353	0.575	24.9	0.463	1.087	25.5	0.608	1.406	26.3	0.803	1.236	26.8	0.904	0.788
	14.7	0.515	0.890	15.0	0.624	1.257	15.3	0.742	1.411	15.6	0.874	1.118	15.8	0.939	0.685

Table 25 (Continued)

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
800	6700	0.000	1.705	6700	0.000	1.488	6700	0.000	1.215	6700	0.001	0.832	6800	0.084	0.589
	4260	0.000	1.613	4260	0.000	1.429	4260	0.000	1.197	4270	0.003	0.872	4370	0.137	0.652
	2340	0.000	1.450	2340	0.000	1.326	2340	0.000	1.169	2350	0.014	0.947	2440	0.227	0.756
	1170	0.000	1.181	1170	0.000	1.159	1170	0.000	1.132	1180	0.050	1.071	1250	0.346	0.906
	391	0.000	0.486	391	0.000	0.759	391	0.009	1.096	404	0.188	1.418	428	0.530	1.193
	217	0.001	-0.072	217	0.005	0.479	219	0.039	1.125	229	0.295	1.651	241	0.618	1.350
-100	9400	0.000	2.042	9400	0.000	1.708	9400	0.000	1.290	9400	0.001	0.705	9560	0.097	0.351
	5220	0.000	2.062	5220	0.000	1.721	5220	0.000	1.294	5220	0.004	0.697	5400	0.194	0.338
	3610	0.000	2.079	3610	0.000	1.732	3610	0.000	1.297	3620	0.013	0.689	3790	0.271	0.328
	1520	0.000	2.136	1520	0.000	1.767	1520	0.000	1.306	1540	0.082	0.662	1640	0.452	0.299
	701	0.000	2.214	701	0.000	1.813	702	0.003	1.312	729	0.220	0.626	776	0.587	0.270
	359	0.000	2.307	359	0.000	1.864	360	0.028	1.311	382	0.366	0.588	403	0.682	0.246
	181	0.000	2.423	182	0.007	1.918	185	0.113	1.299	198	0.507	0.553	206	0.760	0.227
	91.2	0.001	2.550	92.1	0.051	1.961	95.5	0.261	1.278	102	0.633	0.530	105	0.824	0.219
	31.0	0.087	2.615	32.1	0.294	1.950	33.4	0.529	1.265	34.9	0.781	0.556	35.5	0.896	0.249
	14.1	0.433	2.430	14.5	0.587	1.879	14.8	0.733	1.294	15.2	0.877	0.635	15.4	0.941	0.309
-200	8610	0.000	2.093	8610	0.000	1.741	8610	0.000	1.301	8620	0.001	0.686	8800	0.120	0.315
	4530	0.000	2.144	4530	0.000	1.774	4530	0.000	1.311	4540	0.008	0.664	4730	0.246	0.282
	2330	0.000	2.222	2330	0.000	1.823	2330	0.000	1.324	2350	0.043	0.626	2490	0.396	0.238
	1250	0.000	2.328	1250	0.000	1.887	1250	0.000	1.335	1280	0.132	0.573	1370	0.523	0.190
	611	0.000	2.495	611	0.000	1.982	612	0.007	1.342	642	0.286	0.494	682	0.641	0.133
	320	0.000	2.688	320	0.001	2.083	323	0.045	1.331	345	0.431	0.417	362	0.724	0.088
	217	0.000	2.821	217	0.003	2.143	221	0.097	1.313	237	0.510	0.376	247	0.765	0.067

Table 26. Velocity and Temperature Profiles for Liquids at $\mu_o/\mu_w = 2$.

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
± 0.1	5270	0.000	1.861	5270	0.000	1.590	5270	0.000	1.250	5270	0.002	0.775	5290	0.139	0.477
	3350	0.000	1.844	3350	0.000	1.580	3350	0.000	1.248	3350	0.009	0.785	3380	0.220	0.486
	1940	0.000	1.825	1940	0.000	1.568	1940	0.000	1.248	1940	0.033	0.797	1960	0.332	0.492
	1050	0.000	1.804	1050	0.000	1.557	1050	0.000	1.250	1060	0.098	0.811	1070	0.456	0.494
	550	0.000	1.784	550	0.000	1.550	550	0.005	1.256	555	0.216	0.822	562	0.573	0.489
	281	0.000	1.771	282	0.001	1.549	282	0.038	1.268	285	0.363	0.824	288	0.675	0.478
	142	0.000	1.769	142	0.016	1.558	143	0.138	1.282	145	0.511	0.817	146	0.759	0.463
	71.6	0.007	1.783	71.8	0.090	1.579	72.4	0.302	1.294	73.3	0.648	0.802	73.8	0.830	0.445
	24.2	0.227	1.858	24.3	0.407	1.628	24.5	0.600	1.299	24.7	0.811	0.771	24.8	0.910	0.418
	14.6	0.474	1.911	14.7	0.607	1.652	14.8	0.740	1.298	14.9	0.879	0.756	14.9	0.942	0.406
	10.5	0.647	1.946	10.6	0.737	1.667	10.6	0.827	1.296	10.6	0.920	0.746	10.6	0.962	0.398
100	5270	0.000	1.796	5270	0.000	1.548	5270	0.000	1.235	5270	0.002	0.798	5290	0.127	0.523
	3210	0.000	1.755	3210	0.000	1.521	3210	0.000	1.229	3210	0.009	0.818	3230	0.210	0.548
	1640	0.000	1.686	1640	0.000	1.479	1640	0.000	1.219	1640	0.039	0.853	1660	0.342	0.585
	630	0.000	1.558	630	0.000	1.404	630	0.003	1.212	634	0.166	0.919	642	0.524	0.637
	252	0.000	1.389	252	0.002	1.316	252	0.043	1.219	252	0.359	0.995	258	0.668	0.681
	79.1	0.009	1.143	79.3	0.079	1.226	79.8	0.262	1.268	80.8	0.607	1.069	81.4	0.808	0.703
	25.0	0.297	1.232	25.1	0.436	1.330	25.3	0.603	1.318	25.5	0.806	0.997	25.6	0.907	0.609
	15.0	0.548	1.453	15.0	0.648	1.446	15.1	0.759	1.321	15.1	0.885	0.911	15.2	0.945	0.533
	10.7	0.705	1.626	10.7	0.774	1.529	10.7	0.848	1.316	10.8	0.928	0.851	10.8	0.966	0.482
200	5330	0.000	1.738	5330	0.000	1.510	5330	0.000	1.222	5330	0.002	0.819	5360	0.115	0.563
	3390	0.000	1.632	3390	0.000	1.474	3390	0.000	1.212	3390	0.006	0.844	3420	0.184	0.600
	1960	0.000	1.599	1960	0.000	1.422	1970	0.000	1.198	1970	0.024	0.884	1990	0.295	0.650
	1070	0.000	1.481	1070	0.000	1.349	1070	0.000	1.183	1070	0.074	0.942	1080	0.403	0.712
	498	0.000	1.282	498	0.000	1.232	498	0.006	1.169	501	0.191	1.040	508	0.542	0.797
	159	0.000	0.953	159	0.013	1.013	159	0.096	1.137	161	0.434	1.223	163	0.712	0.915
	46.3	0.139	0.595	46.4	0.241	0.948	46.7	0.411	1.235	47.2	0.688	1.282	47.5	0.847	0.886
	20.7	0.475	0.903	20.8	0.564	1.178	20.9	0.684	1.329	21.0	0.842	1.114	21.0	0.924	0.709
	13.4	0.669	1.236	13.4	0.736	1.351	13.4	0.815	1.332	13.5	0.910	0.986	13.5	0.957	0.593

Table 26 (Continued)

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
400	8700	0.000	1.711	8700	0.000	1.491	8700	0.000	1.213	8700	0.000	0.824	8710	0.052	0.582
	3730	0.000	1.568	3730	0.000	1.399	3730	0.000	1.185	3730	0.004	0.884	3750	0.146	0.680
	2270	0.000	1.453	2270	0.000	1.326	2270	0.000	1.164	2270	0.014	0.935	2290	0.227	0.755
	1210	0.000	1.264	1210	0.000	1.207	1210	0.000	1.134	1220	0.048	1.024	1230	0.343	0.866
	488	0.000	0.868	488	0.000	0.969	488	0.005	1.093	491	0.164	1.215	497	0.506	1.058
	222	0.000	0.365	222	0.004	0.693	222	0.043	1.086	225	0.316	1.443	227	0.633	1.236
	122	0.014	-0.079	122	0.038	0.486	123	0.129	1.120	124	0.447	1.616	125	0.716	1.336
800	7380	0.000	1.545	7380	0.000	1.383	7380	0.000	1.174	7380	0.000	0.882	7390	0.050	0.697
	2840	0.000	1.263	2840	0.000	1.201	2840	0.000	1.117	2840	0.005	0.998	2850	0.152	0.891
	1760	0.000	1.059	1760	0.000	1.072	1760	0.000	1.080	1760	0.016	1.088	1770	0.226	1.023
	997	0.000	0.748	997	0.000	0.875	997	0.000	1.029	998	0.047	1.232	1010	0.325	1.210
	534	0.000	0.285	534	0.000	0.593	534	0.003	0.972	537	0.111	1.450	543	0.435	1.435
	315	0.000	-0.235	315	0.001	0.291	315	0.014	0.935	317	0.191	1.691	321	0.525	1.680
-100	7290	0.000	1.929	7290	0.000	1.635	7290	0.000	1.266	7290	0.001	0.749	7320	0.103	0.430
	4050	0.000	1.935	4050	0.000	1.638	4050	0.000	1.268	4050	0.006	0.750	4080	0.203	0.423
	2810	0.000	1.944	2810	0.000	1.645	2810	0.000	1.272	2810	0.017	0.748	2830	0.281	0.414
	1180	0.000	1.990	1180	0.000	1.677	1180	0.000	1.286	1180	0.098	0.734	1200	0.468	0.371
	579	0.000	2.061	579	0.000	1.724	579	0.005	1.304	584	0.236	0.699	592	0.598	0.320
	270	0.000	2.182	270	0.001	1.801	271	0.049	1.323	274	0.411	0.635	277	0.709	0.258
	139	0.000	2.326	139	0.016	1.885	139	0.157	1.327	141	0.551	0.568	143	0.784	0.210

Table 26 (Continued)

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
-100	70.2	0.004	2.495	70.5	0.087	1.966	71.1	0.324	1.309	72.0	0.676	0.508	72.4	0.845	0.177
	23.8	0.167	2.605	24.0	0.379	1.974	24.2	0.596	1.267	24.3	0.815	0.508	24.4	0.912	0.202
	14.4	0.399	2.508	14.5	0.566	1.912	14.6	0.722	1.262	14.7	0.873	0.558	14.7	0.939	0.247
	10.4	0.581	2.392	10.4	0.697	1.854	10.5	0.806	1.267	10.5	0.911	0.603	10.5	0.958	0.285
-200	7380	0.000	1.994	7380	0.000	1.676	7380	0.000	1.280	7380	0.001	0.726	7410	0.115	0.385
	4100	0.000	2.032	4100	0.000	1.701	4100	0.000	1.290	4100	0.007	0.714	4130	0.226	0.355
	2840	0.000	2.068	2840	0.000	1.725	2840	0.000	1.299	2840	0.020	0.701	2870	0.312	0.327
	1270	0.000	2.192	1270	0.000	1.808	1270	0.000	1.327	1280	0.106	0.649	1290	0.495	0.236
	567	0.000	2.400	567	0.000	1.938	567	0.007	1.361	573	0.284	0.545	581	0.643	0.117

Table 27. Velocity and Temperature Profiles for Liquids at $\mu_o/\mu_w = 5$.

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
± 0.1	4720	0.000	1.656	4720	0.000	1.457	4720	0.000	1.205	4720	0.002	0.852	4770	0.106	0.622
	2790	0.000	1.619	2790	0.000	1.434	2790	0.000	1.201	2790	0.008	0.873	2850	0.190	0.643
	1530	0.000	1.582	1530	0.000	1.413	1530	0.000	1.201	1540	0.033	0.899	1580	0.312	0.655
	852	0.000	1.553	852	0.000	1.399	852	0.001	1.207	861	0.096	0.923	892	0.439	0.651
	451	0.000	1.533	451	0.000	1.395	452	0.007	1.223	461	0.214	0.939	478	0.563	0.631
	232	0.000	1.529	233	0.002	1.407	234	0.048	1.250	366	0.366	0.937	249	0.673	0.597
	118	0.000	1.549	118	0.026	1.440	120	0.158	1.280	125	0.521	0.916	128	0.763	0.555
	59.6	0.018	1.602	60.2	0.122	1.494	61.6	0.335	1.306	63.7	0.664	0.881	64.8	0.837	0.511
	20.6	0.314	1.772	20.9	0.473	1.601	21.3	0.644	1.322	21.7	0.831	0.816	21.9	0.919	0.451
	12.7	0.567	1.878	12.8	0.674	1.651	13.0	0.784	1.322	13.1	0.899	0.786	13.2	0.952	0.427
100	4720	0.000	1.573	4720	0.000	1.402	4720	0.000	1.184	4720	0.000	0.879	4770	0.095	0.680
	2790	0.000	1.507	2790	0.000	1.360	2790	0.000	1.173	2790	0.007	0.910	2840	0.172	0.723
	1610	0.000	1.430	1610	0.000	1.312	1610	0.000	1.162	1620	0.025	0.950	1660	0.275	0.769
	877	0.000	1.332	877	0.000	1.253	877	0.000	1.152	884	0.079	1.001	914	0.400	0.819
	458	0.000	1.210	458	0.000	1.182	458	0.006	1.145	467	0.184	1.065	484	0.526	0.873
	209	0.000	1.027	209	0.004	1.080	210	0.054	1.143	217	0.355	1.161	223	0.659	0.939
	70.7	0.026	0.706	71.3	0.106	0.929	72.6	0.279	1.185	75.1	0.603	1.308	76.6	0.804	0.994
	22.0	0.456	0.834	22.2	0.546	1.112	22.5	0.667	1.318	22.9	0.832	1.190	23.1	0.918	0.789
	13.3	0.688	1.188	13.3	0.750	1.328	13.4	0.824	1.348	13.6	0.914	1.035	13.6	0.959	0.635
200	4320	0.000	1.485	4320	0.000	1.344	4320	0.000	1.163	4320	0.002	0.909	4360	0.096	0.741
	2550	0.000	1.390	2550	0.000	1.284	2550	0.000	1.146	2550	0.007	0.951	2600	0.171	0.805
	1480	0.000	1.273	1480	0.000	1.210	1480	0.000	1.126	1480	0.026	1.006	1520	0.270	0.881
	802	0.000	1.113	802	0.000	1.110	802	0.001	1.102	809	0.079	1.082	835	0.392	0.978
	374	0.000	0.854	374	0.000	0.951	375	0.011	1.070	382	0.210	1.211	396	0.537	1.120
	120	0.006	0.277	120	0.036	0.623	121	0.137	1.047	125	0.453	1.505	129	0.716	1.348
	42.1	0.222	-0.174	42.5	0.320	0.488	43.1	0.456	1.192	44.1	0.696	1.651	44.8	0.848	1.306

Table 27 (Continued)

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
400	8660	0.000	1.482	8660	0.000	1.339	8660	0.000	1.156	8660	0.000	0.900	8690	0.030	0.740
	330	0.000	1.280	3330	0.000	1.210	3330	0.000	1.115	3330	0.003	0.981	3370	0.110	0.883
	1970	0.000	1.123	1970	0.000	1.110	1970	0.000	1.084	1970	0.011	1.047	2010	0.186	0.992
	984	0.000	0.854	984	0.000	0.938	984	0.000	1.034	989	0.044	1.164	1020	0.311	1.174
	349	0.000	0.263	349	0.001	0.566	350	0.011	0.938	356	0.178	1.440	368	0.506	1.542
	197	0.003	-0.208	197	0.010	0.281	198	0.048	0.884	203	0.291	1.678	209	0.603	1.786
800	7930	0.000	1.282	7930	0.000	1.206	7930	0.000	1.106	7930	0.000	0.965	7950	0.026	0.876
	2830	0.000	0.945	2830	0.000	0.989	2830	0.000	1.033	2830	0.003	1.095	2860	0.102	1.116
	1320	0.000	0.546	1320	0.000	0.734	1320	0.000	0.952	1320	0.017	1.256	1350	0.207	1.397
	536	0.000	-0.160	536	0.000	0.281	536	0.002	0.816	540	0.079	1.557	557	0.366	1.881
-100	4690	0.000	1.747	4690	0.000	1.517	4690	0.000	1.227	4690	0.002	0.822	4750	0.121	0.559
	2900	0.000	1.745	2900	0.000	1.516	2900	0.000	1.231	2910	0.009	0.831	2970	0.206	0.553
	1650	0.000	1.763	1650	0.000	1.531	1650	0.000	1.244	1650	0.035	0.837	1710	0.331	0.522
	884	0.000	1.813	884	0.000	1.571	884	0.001	1.271	894	0.110	0.829	928	0.473	0.453
	459	0.000	1.912	459	0.000	1.648	459	0.008	1.316	471	0.247	0.785	488	0.604	0.343
	234	0.000	2.081	234	0.002	1.774	235	0.053	1.379	244	0.413	0.681	251	0.711	0.207
	157	0.000	2.217	157	0.009	1.873	159	0.113	1.414	165	0.507	0.593	170	0.763	0.125
-200	6240	0.000	1.841	6240	0.000	1.579	6240	0.000	1.249	6240	0.001	0.788	6300	0.096	0.494
	3300	0.000	1.880	3300	0.000	1.604	3300	0.000	1.263	3300	0.008	0.784	3370	0.210	0.458
	2160	0.000	1.933	2160	0.000	1.641	2160	0.000	1.281	2160	0.024	0.774	2230	0.311	0.404
	2030	0.000	2.026	1340	0.000	1.708	1340	0.000	1.314	1350	0.069	0.751	1400	0.434	0.308
	968	0.000	2.113	968	0.000	1.771	968	0.000	1.345	980	0.122	0.721	1020	0.514	0.220

Table 28. Velocity and Temperature Profiles for Liquids at $\mu_o/\mu_w = 10.$

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
± 0.1	9400	0.000	1.558	9400	0.000	1.391	9400	0.000	1.176	9400	0.000	0.876	9440	0.027	0.690
	3610	0.000	1.488	3610	0.000	1.348	3610	0.000	1.168	3610	0.002	0.915	3690	0.111	0.741
	2140	0.000	1.451	2140	0.000	1.326	2140	0.000	1.166	2140	0.011	0.940	2210	0.198	0.761
	1170	0.000	1.419	1170	0.000	1.309	1170	0.000	1.170	1180	0.042	0.969	1240	0.325	0.766
	618	0.000	1.399	618	0.000	1.304	618	0.002	1.185	632	0.125	0.995	670	0.468	0.746
	336	0.000	1.396	336	0.000	1.314	337	0.016	1.211	351	0.252	1.005	371	0.589	0.707
	175	0.000	1.415	175	0.006	1.346	178	0.076	1.248	188	0.408	0.993	197	0.696	0.653
	89.7	0.003	1.465	90.5	0.050	1.402	93.0	0.210	1.289	98.8	0.562	0.959	102	0.784	0.593
	45.7	0.049	1.556	46.8	0.185	1.481	48.6	0.401	1.322	51.1	0.701	0.911	52.4	0.856	0.535
	16.4	0.438	1.796	16.8	0.571	1.624	17.2	0.711	1.343	17.6	0.864	0.831	17.8	0.935	0.460
100	9400	0.000	1.485	9400	0.000	1.341	9400	0.000	1.157	9400	0.000	0.899	9440	0.024	0.739
	3360	0.000	1.371	3360	0.000	1.270	3360	0.000	1.137	3360	0.002	0.952	3420	0.105	0.822
	1570	0.000	1.259	1570	0.000	1.201	1570	0.000	1.121	1570	0.019	1.007	1630	0.233	0.896
	979	0.000	1.183	979	0.000	1.153	979	0.000	1.110	987	0.050	1.046	1040	0.331	0.944
	534	0.000	1.068	534	0.000	1.082	534	0.003	1.095	546	0.129	1.105	578	0.459	1.015
	178	0.000	0.775	178	0.008	0.898	180	0.067	1.056	190	0.363	1.279	199	0.659	1.187
	57.0	0.109	0.262	57.9	0.197	0.626	59.4	0.350	1.105	62.3	0.633	1.555	64.3	0.816	1.306
	21.2	0.390	-0.082	21.6	0.521	0.619	22.2	0.665	1.324	22.8	0.834	1.593	23.1	0.919	1.154
200	8610	0.000	1.406	8610	0.000	1.289	8610	0.000	1.137	8610	0.000	0.924	8650	0.025	0.792
	3080	0.000	1.252	3080	0.000	1.191	3080	0.000	1.107	3080	0.003	0.990	3130	0.104	0.904
	1440	0.000	1.084	1440	0.000	1.085	1440	0.000	1.076	1440	0.019	1.063	1490	0.228	1.024
	897	0.000	0.957	897	0.000	1.004	897	0.000	1.052	905	0.050	1.118	949	0.321	1.116
	538	0.000	0.788	538	0.000	0.894	539	0.002	1.018	549	0.111	1.195	580	0.426	1.243
	179	0.001	0.233	180	0.010	0.526	181	0.062	0.904	190	0.327	1.501	200	0.627	1.628
	101	0.063	-0.201	101	0.086	0.259	103	0.179	0.874	108	0.464	1.756	113	0.716	1.816

Table 28 (Continued)

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
400	6650	0.000	1.231	6650	0.000	1.172	6650	0.000	1.093	6650	0.000	0.982	6680	0.030	0.911
	2560	0.000	0.999	2560	0.000	1.023	2560	0.000	1.044	2560	0.003	1.072	2610	0.110	1.079
	1510	0.000	0.818	1510	0.000	0.908	1510	0.000	1.006	1510	0.013	1.143	1560	0.187	1.214
	875	0.000	0.577	875	0.000	0.752	875	0.000	0.953	881	0.040	1.237	919	0.285	1.400
	475	0.000	0.220	475	0.000	0.516	475	0.003	0.868	484	0.104	1.385	509	0.403	1.687
	282	0.000	-0.195	282	0.002	0.237	283	0.018	0.767	291	0.188	1.583	307	0.501	2.013
800	7800	0.000	1.023	7800	0.000	1.031	7800	0.000	1.037	7800	0.000	1.046	7820	0.016	1.049
	2130	0.000	0.560	2130	0.000	0.731	2130	0.000	0.932	2130	0.003	1.213	2170	0.102	1.382
	1170	0.000	0.209	1170	0.000	0.505	1170	0.000	0.853	1170	0.015	1.341	1210	0.182	1.647
	780	0.000	-0.100	780	0.000	0.304	780	0.000	0.780	785	0.033	1.454	815	0.249	1.892
-100	9400	0.000	1.635	9400	0.000	1.443	9400	0.000	1.197	9400	0.000	0.853	9450	0.031	0.638
	3610	0.000	1.611	3610	0.000	1.430	3610	0.000	1.200	3620	0.003	0.878	3700	0.127	0.655
	2140	0.000	1.620	2140	0.000	1.437	2140	0.000	1.210	2140	0.013	0.890	2220	0.228	0.636
	1240	0.000	1.659	1240	0.000	1.468	1240	0.000	1.235	1250	0.048	0.897	1320	0.357	0.575
	671	0.000	1.743	671	0.000	1.541	671	0.001	1.286	688	0.137	0.884	732	0.503	0.445
	331	0.000	1.934	331	0.000	1.693	332	0.018	1.387	349	0.305	0.794	370	0.648	0.215
-200	8610	0.000	1.720	8610	0.000	1.500	8610	0.000	1.220	8610	0.000	0.828	8680	0.042	0.582
	4790	0.000	1.734	4790	0.000	1.511	4790	0.000	1.230	4790	0.002	0.835	4870	0.104	0.572
	2530	0.000	1.798	2530	0.000	1.553	2530	0.000	1.255	2540	0.011	0.834	2640	0.228	0.509
	1720	0.000	1.873	1720	0.000	1.606	1720	0.000	1.284	1730	0.030	0.826	1820	0.329	0.424
	1230	0.000	1.968	1230	0.000	1.675	1230	0.000	1.324	1240	0.064	0.811	1320	0.426	0.309

Table 29. Velocity and Temperature Profiles for Liquids at $\mu_o/\mu_w = 20$.

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
± 0.1	9560	0.000	1.411	9560	0.000	1.292	9560	0.000	1.139	9560	0.000	0.923	9600	0.019	0.789
	3410	0.000	1.352	3410	0.000	1.259	3410	0.000	1.135	3420	0.002	0.960	3490	0.092	0.837
	1590	0.000	1.312	1590	0.000	1.236	1590	0.000	1.137	1600	0.016	0.996	1680	0.221	0.859
	1040	0.000	1.298	1040	0.000	1.230	1040	0.000	1.144	1050	0.043	1.016	1120	0.316	0.855
	613	0.000	1.292	613	0.000	1.233	613	0.001	1.160	629	0.109	1.037	679	0.438	0.831
	319	0.000	1.302	319	0.000	1.254	320	0.016	1.192	337	0.242	1.049	364	0.577	0.778
	172	0.000	1.333	172	0.006	1.293	175	0.072	1.234	189	0.393	1.038	201	0.685	0.713
	89.6	0.003	1.395	90.6	0.049	1.358	93.9	0.201	1.282	102	0.549	1.003	107	0.776	0.641
	46.2	0.048	1.497	47.7	0.178	1.448	50.0	0.388	1.326	53.5	0.692	0.950	55.4	0.851	0.570
	17.1	0.425	1.764	17.6	0.558	1.618	18.1	0.700	1.360	18.8	0.858	0.859	19.1	0.932	0.481
100	9560	0.000	1.321	9560	0.000	1.231	9560	0.000	1.114	9560	0.000	0.951	9600	0.017	0.849
	3410	0.000	1.224	3410	0.000	1.172	3410	0.000	1.099	3420	0.002	0.996	3480	0.080	0.923
	1590	0.000	1.122	1590	0.000	1.109	1590	0.000	1.083	1600	0.013	1.045	1670	0.195	0.998
	996	0.000	1.048	996	0.000	1.063	996	0.000	1.070	1010	0.038	1.079	1070	0.290	1.055
	543	0.000	0.932	543	0.000	0.986	544	0.002	1.044	558	0.108	1.132	600	0.421	1.156
	193	0.000	0.610	193	0.006	0.755	195	0.054	0.946	208	0.320	1.326	222	0.621	1.452
	61.8	0.270	-0.102	62.0	0.279	0.357	63.2	0.365	0.969	66.7	0.606	1.748	69.6	0.797	1.674
200	2810	0.000	1.075	2810	0.000	1.072	2810	0.000	1.060	2810	0.002	1.042	2870	0.090	1.025
	1200	0.000	0.888	1200	0.000	0.954	1200	0.000	1.020	1210	0.021	1.115	1270	0.225	1.173
	766	0.000	0.755	766	0.000	0.866	766	0.000	0.987	775	0.053	1.163	824	0.315	1.293
	421	0.000	0.522	421	0.000	0.703	422	0.005	0.914	434	0.132	1.256	466	0.438	1.527
	151	0.013	-0.220	151	0.029	0.160	154	0.091	0.681	163	0.341	1.691	174	0.625	2.183
400	2580	0.000	0.820	2580	0.000	0.899	2580	0.000	0.990	2580	0.002	1.117	2630	0.082	1.197
	1030	0.000	0.483	1030	0.000	0.682	1030	0.000	0.908	1040	0.022	1.228	1090	0.215	1.473
	470	0.000	0.015	470	0.000	0.362	470	0.003	0.764	480	0.090	1.386	511	0.364	1.930
	378	0.000	-0.162	378	0.001	0.235	379	0.007	0.703	389	0.120	1.455	416	0.406	2.111

Table 29 (Continued)

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
800	7930	0.000	0.740	7930	0.000	0.838	7930	0.000	0.961	7930	0.000	1.132	7950	0.011	1.237
	1980	0.000	0.287	1980	0.000	0.540	1980	0.000	0.847	1980	0.003	1.277	2020	0.082	1.560
	1190	0.000	-0.010	1190	0.000	0.347	1190	0.000	0.770	1190	0.010	1.367	1230	0.145	1.796
-100	9560	0.000	1.506	9560	0.000	1.357	9560	0.000	1.165	9560	0.000	0.895	9610	0.022	0.727
	3680	0.000	1.495	3670	0.000	1.356	3680	0.000	1.174	3680	0.002	0.921	3770	0.100	0.741
	2170	0.000	1.511	2170	0.000	1.368	2170	0.000	1.188	2180	0.009	0.935	2270	0.188	0.721
	1200	0.000	1.572	1200	0.000	1.413	1200	0.000	1.223	1210	0.040	0.947	1290	0.331	0.633
	664	0.000	1.692	664	0.000	1.511	664	0.001	1.297	683	0.121	0.944	743	0.485	0.445
	543	0.000	1.749	543	0.000	1.561	544	0.003	1.334	564	0.162	0.932	615	0.535	0.352
-200	5180	0.000	1.627	5180	0.000	1.444	5180	0.000	1.207	5180	0.000	0.876	5270	0.074	0.653
	3210	0.000	1.670	3210	0.000	1.474	3210	0.000	1.227	3210	0.004	0.881	3320	0.146	0.618
	2250	0.000	1.731	2250	0.000	1.514	2250	0.000	1.252	2250	0.011	0.882	2370	0.224	0.555
	1530	0.000	1.842	1530	0.000	1.588	1530	0.000	1.297	1540	0.031	0.879	1660	0.355	0.424

Table 30. Velocity and Temperature Profiles for Liquids at $\mu_o/\mu_w = 0.5$.

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
± 0.1	9050	0.000	2.108	9050	0.000	1.750	9050	0.000	1.303	9050	0.001	0.677	9010	0.129	0.304
	6150	0.000	2.121	6150	0.000	1.759	6150	0.000	1.305	6150	0.004	0.669	6110	0.200	0.297
	3580	0.000	2.142	3580	0.000	1.771	3580	0.000	1.306	3580	0.017	0.656	3540	0.314	0.289
	1340	0.000	2.186	1340	0.000	1.794	1340	0.000	1.304	1330	0.118	0.629	1310	0.504	0.282
	594	0.000	2.225	594	0.000	1.811	594	0.007	1.296	588	0.276	0.614	581	0.628	0.284
	299	0.000	2.255	299	0.001	1.820	298	0.050	1.281	294	0.420	0.610	291	0.713	0.290
	150	0.000	2.277	150	0.015	1.816	149	0.159	1.263	147	0.557	0.615	146	0.786	0.299
	46.1	0.032	2.256	45.9	0.200	1.775	45.5	0.444	1.251	45.0	0.736	0.640	44.7	0.874	0.322
	20.5	0.271	2.174	20.4	0.461	1.736	20.2	0.648	1.254	20.1	0.837	0.665	20.0	0.923	0.341
	13.1	0.499	2.111	13.0	0.634	1.713	13.0	0.762	1.259	12.9	0.890	0.680	12.9	0.948	0.352
100	6990	0.000	2.074	6990	0.000	1.728	6990	0.000	1.295	6990	0.002	0.689	6950	0.162	0.329
	4050	0.000	2.077	4050	0.000	1.729	4050	0.000	1.294	4050	0.011	0.686	4010	0.268	0.330
	2200	0.000	2.076	2200	0.000	1.727	2200	0.000	1.292	2190	0.045	0.684	2170	0.391	0.336
	1150	0.000	2.065	1150	0.000	1.719	1150	0.000	1.287	1140	0.129	0.688	1130	0.508	0.348
	587	0.000	2.037	587	0.000	1.701	587	0.007	1.280	582	0.255	0.702	574	0.611	0.365
	264	0.000	1.975	264	0.002	1.665	264	0.056	1.277	260	0.422	0.729	258	0.713	0.388
	77.3	0.005	1.822	77.1	0.084	1.605	76.5	0.301	1.293	75.5	0.650	0.770	75.0	0.831	0.416
	25.1	0.218	1.828	24.9	0.400	1.617	24.7	0.597	1.290	24.5	0.810	0.757	24.5	0.909	0.407
	14.9	0.467	1.897	14.8	0.602	1.637	14.7	0.738	1.280	14.6	0.878	0.739	14.6	0.942	0.395
	10.5	0.643	1.934	10.5	0.735	1.648	10.5	0.826	1.274	10.4	0.919	0.729	10.4	0.961	0.388
200	7240	0.000	2.036	7240	0.000	1.703	7240	0.000	1.287	7240	0.002	0.705	7200	0.145	0.355
	4720	0.000	2.027	4720	0.000	1.697	4720	0.000	1.285	4720	0.007	0.708	4680	0.222	0.362
	2650	0.000	2.006	2650	0.000	1.683	2650	0.000	1.280	2650	0.027	0.715	2620	0.337	0.378
	961	0.000	1.931	961	0.000	1.637	961	0.000	1.269	956	0.145	0.751	943	0.519	0.418
	376	0.000	1.792	376	0.000	1.559	375	0.022	1.266	371	0.328	0.813	367	0.656	0.464
	109	0.001	1.481	109	0.038	1.437	109	0.202	1.311	107	0.573	0.896	106	0.792	0.512
	32.8	0.152	1.427	32.6	0.318	1.470	32.3	0.523	1.330	32.0	0.770	0.871	31.9	0.890	0.489
	14.4	0.516	1.714	14.4	0.633	1.568	14.3	0.755	1.295	14.2	0.885	0.791	14.2	0.945	0.434

Table 30 (Continued)

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
400	5030	0.000	1.942	5030	0.000	1.642	5030	0.000	1.267	5030	0.004	0.743	4990	0.186	0.420
	3140	0.000	1.899	3140	0.000	1.614	3140	0.000	1.259	3140	0.015	0.762	3110	0.273	0.447
	1800	0.000	1.825	1800	0.000	1.568	1800	0.000	1.249	1790	0.047	0.798	1770	0.379	0.487
	661	0.000	1.603	661	0.000	1.439	661	0.003	1.234	657	0.183	0.907	649	0.547	0.574
	275	0.000	1.267	275	0.002	1.267	275	0.039	1.255	272	0.355	1.038	269	0.669	0.654
	88.7	0.009	0.669	88.5	0.065	1.077	88.0	0.229	1.366	86.9	0.584	1.166	86.2	0.797	0.716
	25.7	0.318	1.009	25.6	0.436	1.293	25.5	0.595	1.359	25.3	0.801	1.002	25.2	0.904	0.591
	14.9	0.558	1.399	14.9	0.653	1.444	14.8	0.762	1.321	14.8	0.886	0.886	14.7	0.945	0.506
	10.5	0.712	1.625	10.5	0.780	1.528	10.5	0.852	1.299	10.4	0.930	0.820	10.4	0.967	0.457
800	7160	0.000	1.849	7160	0.000	1.581	7160	0.000	1.247	7160	0.001	0.779	7140	0.107	0.486
	4670	0.000	1.787	4670	0.000	1.541	4670	0.000	1.235	4670	0.004	0.806	4640	0.167	0.527
	2390	0.000	1.646	2390	0.000	1.452	2390	0.000	1.212	2390	0.020	0.873	2360	0.281	0.609
	986	0.000	1.345	986	0.000	1.272	986	0.000	1.181	983	0.089	1.025	971	0.436	0.745
	399	0.000	0.825	399	0.000	0.985	399	0.011	1.180	396	0.229	1.261	391	0.578	0.903
	127	0.013	-0.188	127	0.032	0.564	127	0.121	1.324	125	0.458	1.565	124	0.727	1.062
-100	8940	0.000	2.147	8940	0.000	1.776	8940	0.000	1.311	8940	0.001	0.661	8890	0.143	0.277
	5630	0.000	2.178	5630	0.000	1.795	5630	0.000	1.316	5630	0.006	0.645	5580	0.236	0.260
	2580	0.000	2.244	2580	0.000	1.834	2580	0.000	1.321	2570	0.043	0.607	2540	0.406	0.234
	1160	0.000	2.334	1160	0.000	1.884	1160	0.001	1.321	1150	0.162	0.558	1140	0.551	0.212
	587	0.000	2.433	587	0.000	1.933	587	0.009	1.309	580	0.306	0.518	573	0.651	0.199
	295	0.000	2.548	295	0.001	1.979	294	0.058	1.279	290	0.450	0.488	287	0.732	0.192
	148	0.000	2.671	148	0.016	2.008	147	0.177	1.238	145	0.583	0.474	144	0.800	0.194
	45.6	0.024	2.744	45.3	0.201	1.950	44.9	0.458	1.199	44.4	0.747	0.508	44.2	0.880	0.228
	20.3	0.243	2.530	20.1	0.452	1.854	20.0	0.648	1.218	19.8	0.839	0.574	19.8	0.924	0.275
	13.0	0.473	2.338	12.9	0.621	1.791	12.8	0.757	1.237	12.8	0.889	0.620	12.7	0.947	0.308

Table 30 (Continued)

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
-200	9730	0.000	2.182	9730	0.000	1.799	9730	0.000	1.319	9730	0.001	0.647	9680	0.139	0.252
	5350	0.000	2.242	5350	0.000	1.837	5350	0.000	1.328	5350	0.008	0.617	5300	0.269	0.218
	2970	0.000	2.320	2970	0.000	1.883	2970	0.000	1.335	2970	0.037	0.573	2930	0.403	0.185
	1570	0.000	2.431	1570	0.000	1.946	1570	0.000	1.339	1570	0.125	0.511	1540	0.528	0.151
	811	0.000	2.582	811	0.000	2.026	810	0.003	1.332	803	0.267	0.442	792	0.633	0.120
	411	0.000	2.771	411	0.000	2.114	411	0.031	1.300	405	0.416	0.380	401	0.718	0.097
	207	0.000	2.985	207	0.005	2.187	206	0.125	1.239	203	0.547	0.338	202	0.784	0.087
	104	0.000	3.188	104	0.044	2.209	103	0.278	1.173	102	0.658	0.328	101	0.837	0.093
	32.0	0.066	3.164	31.8	0.297	2.068	31.5	0.547	1.150	31.2	0.794	0.411	31.1	0.902	0.161
	14.2	0.391	2.665	14.1	0.570	1.898	14.0	0.728	1.203	14.0	0.876	0.537	13.9	0.942	0.248

Table 31. Velocity and Temperature Profiles for Liquids at $\mu_o/\mu_w = 0.2$.

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
± 0.1	6660	0.000	2.249	6660	0.000	1.842	6660	0.000	1.329	6660	0.005	0.612	6510	0.244	0.213
	2920	0.000	2.325	2920	0.000	1.883	2920	0.000	1.331	2910	0.044	0.564	2800	0.419	0.192
	1290	0.000	2.415	7290	0.000	1.930	1290	0.000	1.324	1270	0.166	0.514	1220	0.559	0.183
	648	0.000	2.502	648	0.000	1.969	648	0.008	1.304	629	0.308	0.484	607	0.655	0.183
	325	0.000	2.594	325	0.001	1.998	323	0.054	1.266	311	0.448	0.471	302	0.732	0.189
	163	0.000	2.681	162	0.013	2.003	160	0.167	1.221	153	0.576	0.474	150	0.797	0.202
	50.0	0.018	2.706	49.1	0.185	1.923	47.9	0.444	1.187	46.5	0.740	0.516	45.8	0.877	0.238
	22.0	0.214	2.534	21.5	0.431	1.835	21.1	0.635	1.198	20.6	0.832	0.567	20.4	0.921	0.274
	13.8	0.437	2.364	13.6	0.598	1.779	13.4	0.743	1.212	13.2	0.882	0.603	13.1	0.944	0.299
100	7570	0.000	2.205	7570	0.000	1.813	7570	0.000	1.322	7570	0.003	0.634	7420	0.202	0.240
	3970	0.000	2.243	3970	0.000	1.835	3970	0.000	1.323	3960	0.019	0.609	3830	0.342	0.228
	1460	0.000	2.307	1460	0.000	1.867	1460	0.000	1.317	1440	0.129	0.568	1390	0.523	0.224
	646	0.000	2.355	646	0.000	1.886	645	0.007	1.301	628	0.289	0.552	606	0.640	0.231
	324	0.000	2.383	324	0.001	1.899	323	0.048	1.279	311	0.428	0.553	302	0.720	0.243
	162	0.000	2.392	162	0.013	1.875	160	0.155	1.257	154	0.559	0.565	150	0.788	0.259
	49.9	0.023	2.338	49.1	0.184	1.818	47.9	0.434	1.240	46.5	0.733	0.596	45.8	0.873	0.288
	21.9	0.234	2.232	21.5	0.437	1.771	21.1	0.634	1.230	20.6	0.832	0.619	20.4	0.920	0.308
	13.8	0.457	2.215	13.6	0.607	1.737	13.4	0.747	1.230	13.2	0.884	0.636	13.1	0.945	0.322
200	8060	0.000	2.171	8060	0.000	1.799	8060	0.000	1.315	8060	0.002	0.649	7920	0.178	0.261
	3930	0.000	2.196	3930	0.000	1.806	3930	0.000	1.316	3920	0.017	0.631	3800	0.328	0.255
	1380	0.000	2.222	1380	0.000	1.816	1380	0.000	1.308	1360	0.127	0.611	1310	0.517	0.263
	601	0.000	2.218	601	0.000	1.808	600	0.008	1.296	584	0.286	0.617	564	0.636	0.279
	266	0.000	2.174	266	0.002	1.779	265	0.064	1.287	255	0.450	0.637	248	0.731	0.300
	77.2	0.004	2.019	76.7	0.036	1.719	74.9	0.316	1.293	72.3	0.665	0.673	71.0	0.839	0.332
	24.7	0.205	2.041	24.2	0.400	1.710	23.7	0.604	1.264	23.2	0.816	0.669	23.0	0.912	0.339
	14.4	0.455	2.082	14.2	0.601	1.699	14.0	0.741	1.246	13.8	0.880	0.666	13.7	0.943	0.342
400	8330	0.000	2.115	8330	0.000	1.755	8330	0.000	1.304	8330	0.002	0.673	8200	0.153	0.299
	4920	0.000	2.114	4920	0.000	1.753	4920	0.000	1.302	4920	0.008	0.671	4800	0.273	0.303
	2580	0.000	2.100	2580	0.000	1.743	2580	0.000	1.297	2570	0.037	0.675	2480	0.382	0.317

Table 31 (Continued)

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
400	950	0.000	2.029	950	0.000	1.700	950	0.001	1.288	934	0.169	0.708	899	0.549	0.350
	217	0.000	1.721	217	0.004	1.552	216	0.079	1.317	208	0.460	0.806	202	0.734	0.413
	56.6	0.029	1.334	56.0	0.151	1.497	54.7	0.377	1.381	53.0	0.695	0.836	52.1	0.853	0.437
	17.3	0.404	1.773	17.1	0.548	1.612	16.8	0.700	1.286	16.5	0.860	0.737	16.4	0.933	0.388
800	7580	0.000	2.017	7580	0.000	1.690	7580	0.000	1.284	7580	0.002	0.715	7480	0.142	0.367
	4390	0.000	1.977	4390	0.000	1.665	4390	0.000	1.276	4390	0.008	0.732	4290	0.237	0.392
	2030	0.000	1.879	2030	0.000	1.605	2030	0.000	1.264	2030	0.043	0.781	1960	0.381	0.440
	794	0.000	1.650	794	0.000	1.745	794	0.002	1.255	782	0.162	0.896	753	0.535	0.512
	335	0.000	1.265	335	0.001	1.283	334	0.025	1.292	324	0.324	1.035	314	0.654	0.582
	101	0.007	0.420	101	0.047	1.039	99.2	0.191	1.475	95.6	0.560	1.173	93.4	0.786	0.648
	28.2	0.284	0.908	27.9	0.393	1.327	27.4	0.559	1.404	26.8	0.783	0.957	26.5	0.896	0.529
	16.1	0.511	1.423	15.9	0.614	1.483	15.7	0.735	1.320	15.5	0.874	0.832	15.4	0.940	0.456
	11.2	0.671	1.681	11.1	0.749	1.554	11.0	0.833	1.280	10.9	0.922	0.771	10.9	0.963	0.419
-100	6660	0.000	2.289	6660	0.000	1.868	6660	0.000	1.337	6660	0.006	0.594	6490	0.262	0.187
	2920	0.000	2.393	2920	0.000	1.925	2920	0.000	1.340	2910	0.050	0.529	2800	0.440	0.157
	1290	0.000	2.527	1290	0.000	1.996	1290	0.001	1.332	1270	0.185	0.458	1220	0.579	0.140
	648	0.000	2.666	648	0.000	2.060	648	0.009	1.304	627	0.332	0.411	606	0.671	0.133
	325	0.000	2.825	325	0.001	2.115	323	0.061	1.248	310	0.471	0.384	301	0.745	0.134
	163	0.000	2.991	162	0.014	2.137	160	0.182	1.179	153	0.594	0.380	150	0.806	0.145
	50.0	0.015	3.091	49.1	0.187	2.029	47.8	0.454	1.131	46.4	0.748	0.435	45.8	0.880	0.188
	22.0	0.196	2.808	21.5	0.425	1.903	21.1	0.635	1.163	20.6	0.834	0.513	20.4	0.922	0.239
	13.9	0.417	2.530	13.6	0.589	1.824	13.4	0.739	1.192	13.2	0.881	0.567	13.1	0.944	0.275
-200	6380	0.000	2.340	6380	0.000	1.900	6380	0.000	1.345	6380	0.007	0.569	6200	0.293	0.157
	2550	0.000	2.497	2550	0.000	1.986	2550	0.000	1.349	2530	0.076	0.473	2430	0.488	0.115
	1160	0.000	2.681	1160	0.000	2.083	1160	0.001	1.336	1130	0.231	0.383	1090	0.614	0.091
	589	0.000	2.880	589	0.000	2.173	588	0.015	1.293	567	0.380	0.324	549	0.700	0.079
	280	0.000	3.138	280	0.002	2.257	277	0.090	1.202	265	0.527	0.287	259	0.775	0.078
	78.3	0.001	3.499	77.6	0.091	2.214	75.5	0.369	1.072	72.9	0.708	0.315	71.8	0.861	0.113
	25.0	0.133	3.192	24.4	0.382	1.997	23.8	0.610	1.113	23.3	0.823	0.438	23.1	0.916	0.192
	14.5	0.374	2.751	14.3	0.563	1.881	14.0	0.726	1.166	13.8	0.876	0.522	13.7	0.941	0.246

Table 32. Velocity and Temperature Profiles for Liquids at $\mu_o/\mu_w = 0.1$.

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
± 0.1	8520	0.000	2.301	8520	0.000	1.877	8520	0.000	1.341	8520	0.003	0.590	8210	0.231	0.175
	3320	0.000	2.414	3320	0.000	1.938	3320	0.000	1.343	3300	0.044	0.517	3090	0.435	0.145
	1500	0.000	2.535	1500	0.000	2.001	1500	0.000	1.334	1460	0.168	0.451	1360	0.587	0.133
	757	0.000	2.659	757	0.000	2.058	756	0.006	1.308	719	0.310	0.409	678	0.660	0.130
	381	0.000	2.801	381	0.000	2.107	378	0.045	1.255	354	0.449	0.385	337	0.734	0.133
	191	0.000	2.952	191	0.008	2.130	186	0.153	1.187	174	0.574	0.380	167	0.797	0.142
	50.9	0.014	3.097	49.5	0.186	2.025	47.3	0.462	1.121	45.0	0.752	0.425	44.0	0.882	0.182
	15.4	0.358	2.686	14.9	0.552	1.852	14.5	0.718	1.155	14.1	0.872	0.520	13.9	0.939	0.245
100	7840	0.000	2.278	7840	0.000	1.861	7840	0.000	1.336	7840	0.004	0.600	7550	0.235	0.192
	3210	0.000	2.364	3210	0.000	1.907	3210	0.000	1.336	3190	0.042	0.544	3000	0.424	0.170
	1470	0.000	2.451	1470	0.000	1.952	1470	0.000	1.329	1440	0.156	0.495	1340	0.556	0.163
	751	0.000	2.535	751	0.000	1.990	750	0.005	1.309	716	0.293	0.465	675	0.648	0.164
	379	0.000	2.623	379	0.000	2.018	377	0.041	1.272	353	0.431	0.451	336	0.724	0.169
	191	0.000	2.708	190	0.008	2.027	186	0.141	1.227	174	0.559	0.450	167	0.789	0.180
	50.8	0.017	2.767	49.5	0.184	1.950	47.3	0.453	1.174	45.0	0.746	0.483	44.0	0.879	0.213
	15.3	0.373	2.540	14.9	0.557	1.821	14.4	0.720	1.174	14.1	0.872	0.546	13.9	0.939	0.261
200	7190	0.000	2.255	7190	0.000	1.846	7190	0.000	1.331	7190	0.004	0.610	6920	0.241	0.208
	2950	0.000	2.320	2950	0.000	1.880	2950	0.000	1.330	2930	0.045	0.566	2750	0.425	0.194
	1440	0.000	2.374	1440	0.000	1.907	1440	0.000	1.323	1400	0.148	0.535	1310	0.546	0.193
	711	0.000	2.423	711	0.000	1.927	710	0.006	1.308	678	0.287	0.518	639	0.643	0.197
	353	0.000	2.467	353	0.001	1.935	351	0.042	1.283	329	0.433	0.514	313	0.724	0.206
	108	0.000	2.463	108	0.044	1.907	104	0.253	1.251	97.5	0.632	0.527	94.3	0.824	0.228
	45.1	0.032	2.446	43.8	0.213	1.870	42.0	0.472	1.223	40.0	0.755	0.542	39.1	0.883	0.248
	13.8	0.441	2.380	13.4	0.602	1.778	13.1	0.746	1.192	12.8	0.884	0.577	12.6	0.944	0.281
400	6940	0.000	2.200	6940	0.000	1.810	6940	0.000	1.320	6930	0.004	0.635	6690	0.224	0.243
	3470	0.000	2.221	3470	0.000	1.820	3470	0.000	1.319	3450	0.026	0.619	3270	0.369	0.241
	1240	0.000	2.227	1240	0.000	1.820	1240	0.000	1.311	1210	0.152	0.611	1130	0.544	0.252
	482	0.000	2.181	482	0.000	1.791	481	0.016	1.304	456	0.338	0.629	431	0.671	0.270

Table 32 (Continued)

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
400	140	0.000	1.982	139	0.020	1.719	136	0.175	1.332	127	0.572	0.661	123	0.794	0.297
	41.7	0.058	1.881	40.5	0.233	1.734	39.0	0.471	1.310	37.1	0.751	0.646	36.3	0.882	0.306
	17.7	0.343	2.141	17.2	0.517	1.740	16.7	0.687	1.233	16.3	0.856	0.617	16.0	0.932	0.303
	11.0	0.577	2.171	10.8	0.694	1.712	10.6	0.803	1.211	10.4	0.910	0.621	10.3	0.957	0.312
800	9800	0.000	2.108	9800	0.000	1.749	9800	0.000	1.303	9800	0.001	0.678	9600	0.128	0.303
	5160	0.000	2.094	5160	0.000	1.740	5160	0.000	1.300	5150	0.007	0.683	4960	0.246	0.313
	2390	0.000	2.048	2390	0.000	1.712	2390	0.000	1.293	2370	0.042	0.705	2240	0.395	0.335
	867	0.000	1.898	867	0.000	1.628	867	0.002	1.289	843	0.178	0.778	791	0.558	0.373
	359	0.000	1.627	359	0.000	1.498	358	0.025	1.326	339	0.349	0.861	321	0.674	0.410
	115	0.001	1.013	115	0.034	1.346	112	0.184	1.475	105	0.560	0.925	101	0.788	0.443
	51.8	0.061	0.879	50.9	0.173	1.424	49.1	0.387	1.481	46.6	0.701	0.871	45.3	0.857	0.428
	15.2	0.469	1.803	14.9	0.599	1.633	14.5	0.734	1.269	14.2	0.877	0.693	14.0	0.941	0.354
-100	7950	0.000	2.340	7950	0.000	1.901	7950	0.000	1.347	7940	0.004	0.571	7620	0.263	0.154
	3180	0.000	2.477	3180	0.000	1.975	3180	0.000	1.348	3150	0.054	0.483	2950	0.460	0.119
	1450	0.000	2.638	1450	0.000	2.057	1450	0.000	1.337	1400	0.191	0.402	1310	0.589	0.103
	734	0.000	2.798	734	0.000	2.132	733	0.008	1.301	695	0.337	0.350	656	0.677	0.097
	370	0.000	3.000	370	0.000	2.203	367	0.052	1.229	342	0.480	0.317	326	0.751	0.098
	114	0.000	3.338	114	0.042	2.307	109	0.279	1.098	103	0.658	0.324	99.4	0.838	0.119
	47.9	0.016	3.412	46.5	0.205	2.084	44.4	0.483	1.070	42.3	0.764	0.375	41.4	0.887	0.155
	14.8	0.366	2.803	14.3	0.561	1.874	13.9	0.726	1.138	13.5	0.875	0.500	13.4	0.940	0.233
-200	5480	0.000	2.434	5480	0.000	1.960	5480	0.000	1.355	5460	0.016	0.516	5160	0.369	0.114
	2580	0.000	2.587	2580	0.000	2.037	2580	0.000	1.353	2540	0.094	0.421	2370	0.516	0.086
	1330	0.000	2.707	1330	0.000	2.126	1330	0.001	1.337	1280	0.231	0.342	1200	0.619	0.071
	674	0.000	2.973	674	0.000	2.221	673	0.012	1.285	634	0.379	0.282	600	0.701	0.063
	320	0.000	3.259	320	0.001	2.312	316	0.081	1.177	293	0.525	0.247	281	0.774	0.063
	89.6	0.001	3.713	88.6	0.075	2.278	84.6	0.354	1.019	79.7	0.703	0.271	77.6	0.859	0.093
	26.2	0.111	3.462	25.1	0.375	2.027	24.1	0.611	1.058	23.3	0.826	0.390	22.9	0.918	0.167
	10.9	0.516	2.635	10.6	0.664	1.833	10.4	0.789	1.151	10.2	0.905	0.525	10.1	0.955	0.250

Table 33. Velocity and Temperature Profiles for Liquids at $\mu_o/\mu_w = 0.05$.

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
±0.1	7360	0.000	2.379	7360	0.000	1.925	7360	0.000	1.351	7350	0.007	0.549	6900	0.302	0.134
	4050	0.000	2.469	4050	0.000	1.972	4050	0.000	1.351	4010	0.036	0.489	3690	0.430	0.114
	2130	0.000	2.584	2130	0.000	2.033	2130	0.000	1.346	2070	0.125	0.421	1890	0.542	0.101
	1090	0.000	2.729	1090	0.000	2.104	1090	0.002	1.323	1030	0.263	0.364	950	0.636	0.094
	553	0.000	2.905	553	0.000	2.175	551	0.021	1.269	507	0.405	0.325	473	0.714	0.092
	279	0.000	3.117	278	0.002	2.232	273	0.097	1.182	248	0.536	0.304	234	0.799	0.095
	86.1	0.001	3.445	84.6	0.083	2.192	79.9	0.350	1.069	73.8	0.699	0.316	71.1	0.858	0.115
	35.8	0.047	3.477	34.0	0.285	2.069	32.1	0.545	1.052	30.3	0.795	0.361	29.5	0.904	0.147
	16.7	0.289	3.124	15.9	0.514	1.940	15.3	0.699	1.083	14.7	0.865	0.430	14.4	0.937	0.190
100	7560	0.000	2.348	7560	0.000	1.906	7560	0.000	1.347	7550	0.005	0.565	7110	0.282	0.151
	4100	0.000	2.426	4100	0.000	1.947	4100	0.000	1.347	4080	0.031	0.513	3750	0.413	0.132
	2140	0.000	2.524	2140	0.000	1.998	2140	0.000	1.342	2090	0.113	0.454	1910	0.528	0.121
	1100	0.000	2.642	1100	0.000	2.056	1100	0.001	1.325	1040	0.247	0.405	956	0.625	0.115
	554	0.000	2.781	554	0.000	2.113	552	0.018	1.282	510	0.388	0.372	475	0.704	0.113
	279	0.000	2.944	279	0.002	2.157	274	0.089	1.214	249	0.522	0.353	235	0.772	0.117
	86.1	0.001	3.169	84.7	0.080	2.130	80.1	0.340	1.119	74.0	0.692	0.360	71.2	0.854	0.136
	35.8	0.051	3.236	34.0	0.283	2.035	32.1	0.540	1.090	30.3	0.792	0.392	29.6	0.902	0.162
	16.7	0.299	2.995	15.9	0.516	1.921	15.3	0.699	1.102	14.7	0.865	0.448	14.4	0.936	0.200
200	7110	0.000	2.329	7110	0.000	1.895	7110	0.000	1.343	7110	0.006	0.574	6690	0.283	0.164
	3820	0.000	2.398	3820	0.000	1.929	3820	0.000	1.343	3790	0.034	0.528	3490	0.414	0.148
	1980	0.000	2.480	1980	0.000	1.972	1980	0.000	1.338	1930	0.117	0.478	1760	0.529	0.139
	1010	0.000	2.575	1010	0.000	2.018	1010	0.002	1.322	956	0.250	0.440	879	0.625	0.135
	509	0.000	2.680	509	0.000	2.059	507	0.021	1.287	468	0.390	0.415	436	0.705	0.135
	256	0.000	2.797	256	0.003	2.088	251	0.096	1.236	228	0.524	0.400	216	0.772	0.140
	79.0	0.002	2.952	77.4	0.092	2.063	73.2	0.350	1.164	67.7	0.696	0.404	65.3	0.856	0.158
	32.6	0.074	3.002	31.0	0.308	1.991	29.4	0.556	1.124	27.8	0.799	0.426	27.1	0.905	0.181
	15.1	0.354	2.831	14.5	0.552	1.886	13.9	0.720	1.121	13.4	0.874	0.473	13.2	0.941	0.216
400	6660	0.000	2.285	6660	0.000	1.864	6660	0.000	1.335	6660	0.006	0.595	6280	0.274	0.191
	3170	0.000	2.342	3170	0.000	1.894	3170	0.000	1.334	3140	0.042	0.556	2890	0.428	0.179

Table 33 (Continued)

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u	Pe_L/z	T	u
400	1170	0.000	2.414	1170	0.000	1.930	1170	0.001	1.324	1120	0.196	0.517	1030	0.587	0.175
	517	0.000	2.460	517	0.000	1.946	515	0.017	1.305	478	0.361	0.503	444	0.687	0.178
	69.3	0.005	2.479	67.7	0.110	1.939	64.0	0.369	1.247	59.4	0.704	0.490	57.3	0.859	0.200
	21.1	0.225	2.646	20.1	0.443	1.890	19.2	0.647	1.161	18.4	0.840	0.490	18.0	0.924	0.220
	11.8	0.492	2.561	11.4	0.644	1.814	11.1	0.775	1.148	10.8	0.898	0.521	10.6	0.951	0.246
800	7190	0.000	2.197	7190	0.000	1.807	7190	0.000	1.320	7190	0.004	0.639	6860	0.222	0.243
	3600	0.000	2.207	3600	0.000	1.812	3600	0.000	1.319	3580	0.024	0.629	3320	0.366	0.243
	1280	0.000	2.183	1280	0.000	1.798	1280	0.000	1.315	1250	0.144	0.638	1140	0.540	0.251
	500	0.000	2.084	500	0.000	1.748	498	0.014	1.324	465	0.327	0.666	431	0.666	0.262
	145	0.000	1.762	144	0.017	1.664	140	0.160	1.408	128	0.561	0.682	122	0.790	0.277
	43.1	0.059	1.679	41.6	0.219	1.760	39.5	0.457	1.634	37.0	0.745	0.619	35.9	0.879	0.271
	18.1	0.324	2.222	17.4	0.503	1.396	16.8	0.679	1.218	16.1	0.853	0.560	15.8	0.930	0.261
	11.1	0.553	2.328	10.8	0.680	1.758	10.5	0.796	1.177	10.2	0.907	0.563	10.1	0.956	0.273
-100	7560	0.000	2.405	7560	0.000	1.942	7560	0.000	1.356	7550	0.007	0.536	7070	0.311	0.119
	4100	0.000	2.510	4100	0.000	1.996	4100	0.000	1.355	4070	0.039	0.466	3730	0.442	0.096
	2140	0.000	2.647	2140	0.000	2.069	2140	0.000	1.349	2080	0.136	0.387	1900	0.555	0.081
	1100	0.000	2.821	1100	0.000	2.154	1100	0.002	1.321	1030	0.280	0.321	951	0.648	0.072
	554	0.000	3.046	554	0.000	2.245	552	0.022	1.253	505	0.430	0.273	472	0.727	0.069
	172	0.000	3.479	171	0.014	2.315	165	0.205	1.079	150	0.618	0.255	143	0.819	0.079
	72.7	0.002	3.760	70.3	0.116	2.225	66.2	0.407	1.004	61.5	0.729	0.282	59.5	0.871	0.102
	22.8	0.155	3.499	21.6	0.416	2.018	20.5	0.639	1.039	19.6	0.838	0.376	19.2	0.923	0.158
	12.9	0.409	3.013	12.4	0.598	1.902	11.9	0.752	1.090	11.6	0.888	0.450	11.4	0.946	0.204
-200	4790	0.000	2.524	4790	0.000	2.007	4790	0.000	1.360	4760	0.030	0.461	4360	0.429	0.083
	3030	0.000	2.627	3030	0.000	2.061	3030	0.000	1.357	2970	0.085	0.397	2710	0.513	0.069
	1740	0.000	2.777	1740	0.000	2.141	1740	0.000	1.345	1670	0.196	0.322	1530	0.605	0.056
	600	0.000	3.149	600	0.000	2.300	597	0.021	1.246	547	0.426	0.231	511	0.727	0.047
	259	0.000	3.529	259	0.003	2.393	252	0.130	1.091	229	0.578	0.204	217	0.799	0.051
	68.7	0.002	4.047	66.8	0.130	2.266	62.6	0.429	0.951	58.3	0.741	0.246	56.5	0.877	0.084
	21.2	0.177	3.620	19.9	0.440	2.025	18.9	0.657	1.020	18.1	0.846	0.363	17.8	0.927	0.152
	11.8	0.449	3.006	11.3	0.626	1.895	11.0	0.769	1.089	10.7	0.896	0.452	10.5	0.951	0.205

Table 34. Velocity and Temperature Profiles for Gases at $\mu_o/\mu_w = 1$.

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu
± 0.1	5880	0.000	2.000	5880	0.000	1.680	5880	0.000	1.280	5880	0.003	0.720	5880	0.160	0.380
	3850	0.000	2.000	3850	0.000	1.680	3850	0.000	1.280	3850	0.009	0.720	3850	0.242	0.380
	2380	0.000	2.000	2380	0.000	1.680	2380	0.000	1.280	2380	0.030	0.720	2380	0.342	0.380
	1160	0.000	2.000	1160	0.000	1.680	1160	0.000	1.280	1160	0.110	0.720	1160	0.485	0.380
	435	0.000	2.000	435	0.000	1.680	435	0.015	1.280	435	0.304	0.720	435	0.642	0.380
	193	0.000	2.000	193	0.006	1.680	193	0.097	1.280	193	0.476	0.720	193	0.742	0.380
	97.1	0.001	2.000	97.1	0.048	1.680	97.1	0.240	1.280	97.1	0.611	0.720	97.1	0.812	0.380
	30.0	0.134	2.000	30.0	0.329	1.680	30.0	0.548	1.280	30.0	0.787	0.720	30.0	0.898	0.380
	13.5	0.505	2.000	13.5	0.634	1.680	13.5	0.760	1.280	13.5	0.889	0.720	13.5	0.947	0.380
100	8270	0.000	1.964	8270	0.000	1.658	8270	0.000	1.274	8270	0.001	0.737	8150	0.112	0.403
	4470	0.000	1.947	4470	0.000	1.647	4470	0.000	1.272	4460	0.006	0.747	4340	0.220	0.409
	2460	0.000	1.924	2460	0.000	1.634	2460	0.000	1.271	2450	0.028	0.758	2350	0.345	0.417
	1290	0.000	1.890	1290	0.000	1.614	1290	0.000	1.270	1280	0.094	0.773	1220	0.472	0.428
	664	0.000	1.841	664	0.000	1.589	664	0.003	1.273	646	0.214	0.790	617	0.585	0.442
	337	0.000	1.775	337	0.000	1.559	335	0.030	1.281	321	0.361	0.810	309	0.681	0.458
	169	0.000	1.696	169	0.010	1.533	167	0.121	1.293	159	0.507	0.829	154	0.761	0.470
	85.0	0.004	1.627	84.2	0.072	1.521	82.0	0.282	1.305	78.5	0.641	0.840	76.8	0.828	0.475
	25.4	0.233	1.724	24.9	0.417	1.573	24.3	0.612	1.302	23.7	0.819	0.802	23.5	0.914	0.443
	10.9	0.643	1.891	10.7	0.736	1.636	10.6	0.828	1.288	10.5	0.921	0.754	10.5	0.962	0.406
200	7900	0.000	1.924	7900	0.000	1.632	7900	0.000	1.266	7900	0.001	0.753	7780	0.111	0.431
	4580	0.000	1.893	4580	0.000	1.735	4580	0.000	1.261	4580	0.005	0.768	4460	0.201	0.447
	2360	0.000	1.841	2360	0.000	1.580	2360	0.000	1.255	2350	0.027	0.794	2260	0.335	0.470
	1260	0.000	1.771	1260	0.000	1.540	1260	0.000	1.250	1250	0.087	0.827	1190	0.456	0.498
	656	0.000	1.671	656	0.000	1.484	656	0.003	1.251	640	0.198	0.870	611	0.569	0.530
	335	0.000	1.531	335	0.000	1.416	333	0.027	1.263	320	0.340	0.921	308	0.666	0.563
	169	0.000	1.354	169	0.010	1.347	167	0.110	1.291	159	0.490	0.970	154	0.751	0.591
	44.9	0.081	1.196	44.0	0.232	1.356	42.8	0.453	1.342	41.4	0.735	0.966	40.8	0.873	0.569
	13.5	0.371	1.674	13.3	0.677	1.552	13.2	0.785	1.305	13.0	0.900	0.818	12.9	0.952	0.455
400	7900	0.000	1.853	7900	0.000	1.586	7900	0.000	1.250	7900	0.001	0.780	7800	0.098	0.481
	4580	0.000	1.795	4580	0.000	1.549	4580	0.000	1.240	4580	0.004	0.807	4480	0.178	0.515

Table 34 (Continued)

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu
400	2360	0.000	1.696	2360	0.000	1.487	2360	0.000	1.226	2350	0.022	0.856	2270	0.304	0.567
	1260	0.000	1.565	1260	0.000	1.409	1260	0.000	1.214	1250	0.071	0.921	1200	0.423	0.624
	586	0.000	1.332	586	0.000	1.278	586	0.004	1.209	572	0.189	1.028	546	0.557	0.703
	174	0.000	0.761	174	0.010	1.018	172	0.089	1.274	165	0.444	1.222	159	0.724	0.815
	51.1	0.114	0.477	51.4	0.206	1.011	50.3	0.393	1.376	48.5	0.694	1.224	47.6	0.853	0.776
	14.1	0.612	1.379	13.9	0.699	1.431	13.8	0.793	1.327	13.6	0.902	0.910	13.5	0.953	0.527
800	8270	0.000	1.740	8270	0.000	1.512	8270	0.000	1.225	8270	0.001	0.822	8190	0.075	0.562
	4710	0.000	1.636	4710	0.000	1.446	4710	0.000	1.206	4700	0.003	0.869	4620	0.143	0.629
	2390	0.000	1.459	2390	0.000	1.334	2390	0.000	1.178	2380	0.015	0.954	2310	0.255	0.729
	1140	0.000	1.176	1140	0.000	1.162	1140	0.000	1.146	1130	0.061	1.095	1090	0.393	0.862
	427	0.000	0.583	427	0.000	0.830	427	0.008	1.133	416	0.201	1.371	398	0.558	1.062
	190	0.002	-0.152	189	0.010	0.486	188	0.059	1.205	181	0.366	1.633	174	0.676	1.215
-100	9910	0.000	2.047	9910	0.000	1.711	9910	0.000	1.292	9910	0.001	0.705	9780	0.105	0.344
	6100	0.000	2.065	6100	0.000	1.723	6100	0.000	1.296	6100	0.003	0.698	5950	0.190	0.329
	2700	0.000	2.109	2700	0.000	1.752	2700	0.000	1.305	2690	0.031	0.677	2580	0.373	0.297
	1280	0.000	2.172	1280	0.000	1.791	1280	0.000	1.315	1260	0.126	0.639	1200	0.524	0.267
	660	0.000	2.251	660	0.000	1.838	660	0.005	1.321	638	0.267	0.594	610	0.633	0.242
	336	0.000	2.358	336	0.000	1.897	334	0.040	1.314	318	0.422	0.547	307	0.721	0.221
	159	0.000	2.501	159	0.013	1.959	156	0.161	1.278	148	0.571	0.510	144	0.796	0.208
	82.4	0.002	2.630	81.6	0.076	1.982	79.0	0.325	1.236	75.7	0.683	0.499	74.2	0.849	0.211
	25.4	0.159	2.593	24.7	0.391	1.887	24.0	0.612	1.222	23.5	0.824	0.560	23.2	0.917	0.263
	10.9	0.575	2.267	10.7	0.696	1.776	10.6	0.806	1.255	10.5	0.911	0.646	10.4	0.958	0.325
-200	7720	0.000	2.113	7720	0.000	1.754	7720	0.000	1.306	7720	0.002	0.679	7560	0.162	0.297
	4300	0.000	2.166	4300	0.000	1.789	4300	0.000	1.317	4290	0.011	0.655	4140	0.295	0.259
	2410	0.000	2.241	2410	0.000	1.836	2410	0.000	1.329	2390	0.049	0.616	2280	0.432	0.216
	1210	0.000	2.367	1210	0.000	1.912	1210	0.000	1.342	1180	0.163	0.544	1120	0.568	0.164
	641	0.000	2.524	641	0.000	2.001	640	0.007	1.346	615	0.314	0.464	590	0.668	0.120
	312	0.000	2.750	312	0.001	2.116	309	0.057	1.320	293	0.477	0.379	284	0.753	0.081
	163	0.000	2.984	163	0.012	2.210	159	0.176	1.260	151	0.604	0.323	147	0.814	0.062
	44.8	0.020	3.291	43.7	0.213	2.178	42.2	0.495	1.154	40.8	0.772	0.345	40.3	0.892	0.104
	13.7	0.411	2.700	13.4	0.591	1.924	13.2	0.745	1.213	13.0	0.884	0.531	12.9	0.945	0.241

Table 35. Velocity and Temperature Profiles for Gases at $\mu_o/\mu_w = 1.5$.

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu
± 0.1	5850	0.000	1.921	5850	0.000	1.625	5850	0.000	1.255	5860	0.002	0.738	6070	0.115	0.442
	2800	0.000	1.897	2800	0.000	1.608	2800	0.000	1.247	2810	0.014	0.744	3010	0.225	0.470
	1290	0.000	1.865	1290	0.000	1.585	1290	0.000	1.234	1310	0.064	0.763	1450	0.361	0.504
	584	0.000	1.821	584	0.000	1.552	585	0.004	1.218	617	0.171	0.797	696	0.498	0.537
	213	0.000	1.744	213	0.003	1.497	217	0.058	1.212	240	0.361	0.852	271	0.647	0.563
	82.2	0.003	1.644	83.6	0.057	1.451	88.1	0.218	1.240	100	0.547	0.888	111	0.765	0.562
	27.1	0.179	1.593	28.5	0.327	1.477	30.7	0.510	1.283	34.1	0.746	0.873	36.4	0.873	0.517
	13.8	0.515	1.704	14.4	0.618	1.547	15.2	0.733	1.294	16.3	0.868	0.821	17.0	0.936	0.462
100	6060	0.000	1.875	6060	0.000	1.595	6060	0.000	1.244	6060	0.002	0.754	6260	0.102	0.472
	2840	0.000	1.824	2840	0.000	1.560	2840	0.000	1.229	2850	0.012	0.770	3040	0.208	0.520
	1380	0.000	1.753	1380	0.000	1.512	1380	0.000	1.209	1400	0.051	0.801	1540	0.329	0.580
	602	0.000	1.633	602	0.000	1.431	603	0.003	1.180	632	0.151	0.867	710	0.470	0.663
	215	0.000	1.402	215	0.003	1.285	219	0.053	1.155	241	0.335	0.990	271	0.624	0.770
	94.1	0.004	1.134	95.4	0.046	1.144	99.7	0.181	1.167	112	0.499	1.103	125	0.734	0.846
	28.9	0.252	0.894	30.1	0.362	1.078	32.0	0.514	1.233	35.3	0.736	1.168	37.8	0.866	0.830
	14.7	0.614	1.068	15.1	0.682	1.218	15.8	0.767	1.288	16.7	0.881	1.069	17.3	0.941	0.691
200	6280	0.000	1.834	6280	0.000	1.568	6280	0.000	1.235	6290	0.001	0.768	6470	0.092	0.499
	2710	0.000	1.751	2710	0.000	1.512	2710	0.000	1.212	2720	0.012	0.796	2890	0.203	0.570
	1350	0.000	1.645	1350	0.000	1.441	1350	0.000	1.185	1370	0.048	0.840	1500	0.316	0.653
	596	0.000	1.461	596	0.000	1.320	597	0.003	1.144	623	0.140	0.931	698	0.452	0.777
	185	0.000	1.021	185	0.007	1.051	188	0.065	1.099	207	0.345	1.145	233	0.630	0.988
	54.4	0.085	0.404	55.9	0.171	0.755	58.9	0.319	1.132	65.9	0.598	1.391	72.2	0.790	1.149
	17.8	0.489	0.161	18.6	0.586	0.745	19.5	0.699	1.271	21.0	0.844	1.422	21.9	0.923	1.029
	13.2	0.498	-0.008	13.8	0.615	0.685	14.7	0.740	1.308	15.7	0.874	1.468	16.3	0.939	1.038
400	6520	0.000	1.760	6520	0.000	1.520	6520	0.000	1.217	6530	0.001	0.793	6680	0.077	0.548
	2750	0.000	1.629	2750	0.000	1.432	2750	0.000	1.183	2760	0.009	0.838	2910	0.180	0.652
	1280	0.000	1.443	1280	0.000	1.310	1280	0.000	1.139	1290	0.044	0.913	1410	0.298	0.791
	653	0.000	1.202	653	0.000	1.152	653	0.002	1.089	675	0.109	1.023	750	0.406	0.952
	204	0.000	0.510	205	0.005	0.732	207	0.048	1.006	225	0.293	1.342	253	0.586	1.311
	107	0.016	-0.028	108	0.048	0.446	111	0.141	0.995	123	0.419	1.568	137	0.676	1.512

Table 35 (Continued)

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu
800	5850	0.000	1.599	5850	0.000	1.419	5850	0.000	1.180	5860	0.001	0.845	5980	0.071	0.652
	2620	0.000	1.410	2620	0.000	1.287	2620	0.000	1.131	2630	0.007	0.914	2760	0.156	0.801
	1320	0.000	1.147	1320	0.000	1.114	1320	0.000	1.070	1340	0.031	1.016	1440	0.252	0.990
	561	0.000	0.631	561	0.000	0.783	561	0.003	0.970	579	0.102	1.239	639	0.383	1.326
	210	0.003	-0.345	210	0.008	0.198	212	0.039	0.853	227	0.241	1.664	254	0.534	1.844
-100	8900	0.000	1.973	8900	0.000	1.661	8900	0.000	1.269	8900	0.001	0.722	9110	0.076	0.401
	4860	0.000	1.975	4860	0.000	1.661	4860	0.000	1.268	4860	0.004	0.719	5100	0.152	0.408
	2400	0.000	1.985	2400	0.000	1.665	2400	0.000	1.266	2420	0.023	0.714	2630	0.271	0.413
	1270	0.000	2.003	1270	0.000	1.675	1270	0.000	1.265	1300	0.074	0.713	1450	0.389	0.410
	650	0.000	2.037	650	0.000	1.692	651	0.003	1.264	687	0.172	0.713	778	0.506	0.393
	330	0.000	2.091	330	0.000	1.719	332	0.025	1.267	364	0.301	0.704	413	0.612	0.357
	148	0.000	2.186	148	0.011	1.766	153	0.114	1.291	174	0.463	0.669	194	0.720	0.292
	53.6	0.012	2.314	55.4	0.116	1.862	59.5	0.323	1.330	67.7	0.641	0.587	73.9	0.820	0.204
	24.5	0.143	2.429	26.0	0.316	1.953	28.2	0.520	1.333	31.5	0.761	0.517	33.6	0.882	0.157
	12.8	0.431	2.536	13.6	0.567	1.985	14.5	0.709	1.302	15.6	0.861	0.491	16.3	0.932	0.162
-200	9380	0.000	2.018	9380	0.000	1.690	9380	0.000	1.280	9380	0.001	0.707	9610	0.078	0.371
	5000	0.000	2.041	5000	0.000	1.704	5000	0.000	1.284	5010	0.004	0.696	5260	0.162	0.363
	2580	0.000	2.085	2580	0.000	1.732	2580	0.000	1.290	2600	0.023	0.679	2840	0.282	0.343
	1240	0.000	2.169	1240	0.000	1.783	1240	0.000	1.302	1270	0.090	0.653	1430	0.425	0.296
	607	0.000	2.301	607	0.000	1.862	608	0.004	1.316	649	0.210	0.613	740	0.553	0.215
	387	0.000	2.415	387	0.000	1.928	389	0.019	1.329	428	0.302	0.571	488	0.625	0.147

Table 36. Velocity and Temperature Profiles for Gases at $\mu_o/\mu_w = 2.0$.

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu
± 0.1	6560	0.000	1.883	6560	0.000	1.599	6560	0.000	1.243	6560	0.001	0.746	6770	0.085	0.465
	2770	0.000	1.846	2770	0.000	1.571	2770	0.000	1.228	2780	0.012	0.753	2990	0.194	0.510
	1360	0.000	1.803	1360	0.000	1.540	1360	0.000	1.211	1390	0.049	0.772	1550	0.307	0.557
	602	0.000	1.737	602	0.000	1.682	603	0.003	1.184	635	0.143	0.815	736	0.441	0.617
	216	0.000	1.623	216	0.003	1.406	220	0.051	1.163	247	0.317	0.893	292	0.594	0.678
	94.6	0.002	1.496	95.9	0.039	1.334	109	0.169	1.178	118	0.475	0.953	140	0.707	0.705
	28.6	0.169	1.387	30.3	0.298	1.335	33.2	0.463	1.237	39.3	0.699	0.971	44.9	0.841	0.655
	14.8	0.490	1.495	15.8	0.582	1.419	17.2	0.693	1.272	19.7	0.838	0.915	21.5	0.918	0.566
100	6320	0.000	1.830	6320	0.000	1.564	6320	0.000	1.230	6320	0.001	0.763	6510	0.081	0.501
	2910	0.000	1.771	2910	0.000	1.522	2910	0.000	1.210	2920	0.009	0.777	3110	0.173	0.557
	1310	0.000	1.680	1310	0.000	1.458	1310	0.000	1.180	1330	0.046	0.811	1480	0.294	0.642
	592	0.000	1.542	592	0.000	1.362	592	0.003	1.137	622	0.132	0.879	715	0.421	0.752
	199	0.000	1.237	199	0.005	1.156	203	0.054	1.078	227	0.313	1.032	267	0.587	0.940
	55.3	0.050	0.683	57.3	0.141	0.851	61.3	0.293	1.061	71.8	0.563	1.271	84.0	0.759	1.167
	19.9	0.556	0.374	20.7	0.605	0.737	21.9	0.682	1.131	24.7	0.815	1.403	27.2	0.903	1.161
200	6560	0.000	1.785	6560	0.000	1.534	6560	0.000	1.219	6560	0.001	0.778	6730	0.071	0.530
	2960	0.000	1.704	2960	0.000	1.477	2960	0.000	1.193	2970	0.008	0.799	3140	0.159	0.602
	1320	0.000	1.574	1320	0.000	1.387	1320	0.000	1.153	1340	0.041	0.844	1480	0.276	0.714
	628	0.000	1.388	628	0.000	1.261	629	0.002	1.101	655	0.113	0.926	748	0.393	0.857
	203	0.000	0.918	203	0.005	0.952	206	0.049	1.008	229	0.291	1.142	268	0.565	1.156
	52.0	0.025	0.223	52.6	0.276	0.548	54.9	0.359	0.956	62.6	0.573	1.454	72.5	0.756	1.485
	39.8	0.205	-0.183	41.3	0.286	0.333	44.1	0.408	0.948	51.0	0.629	1.632	58.5	0.790	1.632
400	6100	0.000	1.689	6100	0.000	1.470	6100	0.000	1.194	6100	0.001	0.809	6250	0.066	0.594
	2680	0.000	1.559	2680	0.000	1.381	2680	0.000	1.156	2690	0.008	0.845	2840	0.152	0.699
	1260	0.000	1.369	1260	0.000	1.251	1260	0.000	1.102	1280	0.037	0.910	1400	0.257	0.852
	652	0.000	1.116	652	0.000	1.081	652	0.002	1.036	675	0.094	1.012	760	0.356	1.041
	295	0.000	0.656	295	0.001	0.780	297	0.021	0.940	319	0.200	1.208	368	0.474	1.345
	128	0.007	-0.104	129	0.029	0.311	132	0.098	0.818	148	0.344	1.523	173	0.603	1.799

Table 36 (Continued)

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu
800	6100	0.000	1.534	6100	0.000	1.366	6100	0.000	1.155	6100	0.001	0.859	6210	0.052	0.696
	2860	0.000	1.361	2860	0.000	1.250	2860	0.000	1.107	2860	0.005	0.909	2990	0.117	0.827
	1300	0.000	1.068	1300	0.000	1.052	1300	0.000	1.028	1310	0.025	1.005	1420	0.215	1.051
	662	0.000	0.674	662	0.000	0.789	662	0.001	0.929	679	0.071	1.152	754	0.311	1.338
	243	0.001	-0.287	244	0.004	0.172	245	0.025	0.734	262	0.189	1.528	301	0.459	1.971
-100	6560	0.000	1.938	6560	0.000	1.636	6560	0.000	1.257	6570	0.001	0.728	6790	0.094	0.429
	2770	0.000	1.934	2770	0.000	1.630	2770	0.000	1.251	2780	0.014	0.725	3010	0.212	0.451
	1360	0.000	1.938	1360	0.000	1.630	1360	0.000	1.244	1390	0.056	0.729	1570	0.330	0.465
	678	0.000	1.956	678	0.000	1.636	678	0.002	1.239	715	0.141	0.741	832	0.448	0.462
	300	0.000	2.004	300	0.001	1.658	303	0.027	1.244	337	0.283	0.751	401	0.578	0.421
	108	0.000	2.118	109	0.025	1.727	114	0.150	1.306	134	0.474	0.713	160	0.710	0.295
	55.2	0.011	2.224	57.0	0.099	1.834	61.8	0.283	1.373	74.1	0.591	0.636	86.8	0.782	0.167
-200	5890	0.000	2.004	5890	0.000	1.679	5890	0.000	1.274	5890	0.002	0.706	6160	0.119	0.388
	2810	0.000	2.037	2810	0.000	1.699	2810	0.000	1.277	2830	0.016	0.693	3090	0.232	0.382
	1290	0.000	2.107	1290	0.000	1.742	1290	0.000	1.286	1330	0.071	0.679	1520	0.371	0.348
	659	0.000	2.216	659	0.000	1.808	660	0.003	1.302	703	0.167	0.660	831	0.490	0.275
	443	0.000	2.310	443	0.000	1.865	444	0.011	1.319	487	0.239	0.639	582	0.556	0.204

Table 37. Velocity and Temperature Profiles for Gases at $\mu_o/\mu_w = 0.667$.

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu
± 0.1	4700	0.000	2.135	4700	0.000	1.773	4700	0.000	1.318	4670	0.012	0.676	3920	0.349	0.252
	1730	0.000	2.196	1730	0.000	1.813	1730	0.000	1.334	1610	0.119	0.616	1320	0.555	0.228
	762	0.000	2.267	762	0.000	1.861	760	0.004	1.349	645	0.313	0.548	554	0.677	0.223
	360	0.000	2.356	360	0.000	1.920	350	0.049	1.330	282	0.484	0.516	254	0.761	0.226
	165	0.000	2.483	163	0.016	1.982	147	0.207	1.243	122	0.629	0.512	114	0.829	0.238
	83.6	0.002	2.633	78.9	0.103	1.964	68.2	0.398	1.179	59.8	0.730	0.531	56.9	0.873	0.257
	23.0	0.231	2.569	20.5	0.483	1.806	18.9	0.684	1.210	17.8	0.860	0.612	17.4	0.935	0.309
100	4460	0.000	2.066	4460	0.000	1.729	4460	0.000	1.307	4430	0.012	0.709	3740	0.336	0.293
	1690	0.000	2.073	1690	0.000	1.738	1690	0.000	1.320	1590	0.106	0.683	1300	0.536	0.282
	756	0.000	2.075	756	0.000	1.750	754	0.004	1.341	648	0.286	0.641	553	0.660	0.286
	336	0.000	2.080	336	0.001	1.777	327	0.049	1.348	265	0.471	0.622	238	0.754	0.296
	159	0.000	2.110	158	0.017	1.819	143	0.196	1.299	119	0.614	0.621	110	0.821	0.308
	82.2	0.003	2.201	77.6	0.103	1.829	67.5	0.384	1.248	59.0	0.720	0.629	56.1	0.869	0.319
	22.6	0.257	2.333	20.3	0.490	1.756	18.8	0.684	1.238	17.7	0.859	0.656	17.3	0.935	0.339
200	8080	0.000	2.016	8080	0.000	1.695	8080	0.000	1.514	8070	0.002	0.729	7320	0.175	0.345
	4410	0.000	2.000	4410	0.000	1.687	4410	0.000	1.295	4380	0.010	0.741	3720	0.316	0.334
	1690	0.000	1.958	1690	0.000	1.668	1690	0.000	1.306	1600	0.092	0.746	1300	0.517	0.334
	754	0.000	1.896	754	0.000	1.647	753	0.003	1.331	654	0.261	0.731	555	0.644	0.348
	336	0.000	1.813	336	0.001	1.634	328	0.042	1.363	268	0.447	0.727	239	0.741	0.366
	169	0.000	1.747	168	0.013	1.652	154	0.162	1.357	127	0.589	0.731	118	0.808	0.379
	51.2	0.038	1.912	46.1	0.231	1.698	40.9	0.493	1.289	36.8	0.771	0.716	35.4	0.893	0.377
	18.1	0.374	2.126	16.7	0.562	1.707	15.7	0.726	1.262	14.9	0.877	0.697	14.6	0.943	0.366
400	7720	0.000	1.934	7720	0.000	1.642	7720	0.000	1.277	7710	0.002	0.766	7050	0.159	0.403
	3910	0.000	1.874	3910	0.000	1.607	3910	0.000	1.274	3890	0.011	0.801	3320	0.308	0.409
	1500	0.000	1.741	1500	0.000	1.537	1500	0.000	1.282	1420	0.089	0.858	1160	0.505	0.434
	670	0.000	1.561	670	0.000	1.454	669	0.004	1.316	585	0.248	0.893	496	0.631	0.467

Table 37 (Continued)

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu
400	338	0.000	1.352	338	0.001	1.382	332	0.032	1.376	275	0.404	0.921	242	0.718	0.496
	170	0.000	1.113	169	0.012	1.351	157	0.134	1.435	129	0.554	0.941	119	0.790	0.515
	50.4	0.066	1.243	46.2	0.230	1.511	41.3	0.471	1.381	37.1	0.756	0.863	35.5	0.886	0.473
	17.8	0.415	1.867	16.6	0.578	1.636	15.7	0.730	1.292	14.9	0.878	0.754	14.6	0.943	0.406
800	6950	0.000	1.778	6950	0.000	1.543	6950	0.000	1.247	6940	0.001	0.833	6410	0.143	0.511
	4250	0.000	1.687	4250	0.000	1.487	4250	0.000	1.238	4230	0.006	0.884	3730	0.236	0.540
	2150	0.000	1.514	2150	0.000	1.387	2150	0.000	1.229	2110	0.030	0.980	1770	0.382	0.585
	776	0.000	1.103	776	0.000	1.171	775	0.001	1.254	709	0.161	1.151	589	0.564	0.672
	340	0.000	0.581	340	0.001	0.945	336	0.021	1.363	285	0.337	1.275	247	0.679	0.743
	170	0.002	0.033	169	0.013	0.807	161	0.098	1.522	133	0.494	1.338	120	0.760	0.780
	135	0.013	-0.060	134	0.029	0.816	126	0.144	1.556	105	0.537	1.329	95.6	0.781	0.773
-100	4970	0.000	2.208	4970	0.000	1.819	4970	0.000	1.330	4040	0.012	0.641	4110	0.366	0.210
	1760	0.000	2.334	1760	0.000	1.895	1760	0.000	1.348	1630	0.138	0.540	1330	0.577	0.172
	769	0.000	2.482	769	0.000	1.983	766	0.005	1.353	641	0.345	0.448	555	0.696	0.157
	361	0.000	2.660	361	0.001	2.077	349	0.059	1.301	280	0.513	0.404	254	0.777	0.155
	165	0.000	2.884	163	0.017	2.151	145	0.230	1.175	121	0.649	0.401	113	0.839	0.168
	83.8	0.001	3.085	78.8	0.108	2.097	67.7	0.417	1.106	59.6	0.742	0.433	56.9	0.879	0.194
	23.2	0.210	2.830	20.5	0.479	1.858	18.9	0.684	1.180	17.8	0.861	0.565	17.4	0.936	0.278
-200	7390	0.000	2.240	7390	0.000	1.839	7390	0.000	1.335	7370	0.004	0.628	6340	0.285	0.191
	3140	0.000	2.374	3140	0.000	1.920	3140	0.000	1.352	3040	0.055	0.542	2450	0.501	0.135
	1370	0.000	2.563	1370	0.000	2.030	1370	0.000	1.364	1200	0.237	0.409	1010	0.646	0.101
	643	0.000	2.795	643	0.000	2.154	638	0.014	1.340	517	0.429	0.321	457	0.740	0.083
	312	0.000	3.064	312	0.001	2.273	294	0.102	1.226	236	0.574	0.283	217	0.805	0.082
	163	0.000	3.324	161	0.020	2.326	142	0.259	1.095	119	0.673	0.288	112	0.850	0.097
	83.3	0.001	3.563	78.0	0.116	2.225	66.7	0.438	1.030	59.0	0.753	0.334	56.4	0.884	0.132
	23.4	0.190	3.118	20.5	0.476	1.912	18.9	0.686	1.147	17.8	0.863	0.516	17.3	0.937	0.245

Table 38. Velocity and Temperature Profiles for Gases at $\mu_o/\mu_w = 0.5$.

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu
± 0.1	5520	0.000	2.268	5520	0.000	1.859	5520	0.000	1.348	5380	0.019	0.615	3550	0.457	0.151
	2580	0.000	2.360	2580	0.000	1.918	2580	0.000	1.367	2220	0.124	0.506	1520	0.602	0.139
	825	0.000	2.560	825	0.000	2.047	817	0.007	1.391	542	0.428	0.372	452	0.743	0.139
	373	0.000	2.773	373	0.001	2.179	330	0.097	1.256	222	0.586	0.356	198	0.813	0.150
	167	0.000	3.087	161	0.028	2.270	120	0.317	1.051	93.1	0.709	0.378	86.8	0.867	0.172
	84.3	0.002	3.467	68.7	0.174	2.050	52.9	0.498	1.016	45.3	0.787	0.427	43.2	0.902	0.203
	39.2	0.060	3.645	28.3	0.405	1.864	24.3	0.652	1.088	22.2	0.851	0.509	21.5	0.931	0.250
	13.6	0.470	2.587	12.2	0.657	1.797	11.4	0.793	1.208	10.8	0.909	0.614	10.6	0.958	0.311
100	8270	0.000	2.168	8270	0.000	1.798	8270	0.000	1.332	8220	0.004	0.673	5860	0.326	0.198
	3290	0.000	2.229	3290	0.000	1.837	3290	0.000	1.351	3030	0.061	0.611	2010	0.535	0.174
	1490	0.000	2.290	1490	0.000	1.888	1490	0.000	1.383	1130	0.248	0.495	850	0.661	0.171
	712	0.000	2.375	712	0.000	1.957	702	0.010	1.409	466	0.434	0.439	389	0.747	0.177
	328	0.000	2.516	327	0.001	2.074	286	0.108	1.297	194	0.590	0.426	174	0.815	1.190
	167	0.000	2.726	162	0.025	2.170	122	0.294	1.133	93.6	0.695	0.439	86.9	0.861	0.206
	84.2	0.002	3.096	69.3	0.164	2.019	53.4	0.484	1.073	45.4	0.779	0.472	43.2	0.899	0.229
	38.9	0.066	3.391	28.4	0.400	1.850	24.4	0.647	1.114	22.2	0.848	0.534	21.5	0.930	0.265
	13.5	0.475	2.536	12.1	0.658	1.789	11.4	0.793	1.214	10.8	0.909	0.623	10.6	0.958	0.317
200	7890	0.000	2.116	7890	0.000	1.766	7890	0.000	1.324	7850	0.004	0.700	5650	0.314	0.226
	3230	0.000	2.139	3230	0.000	1.784	3230	0.000	1.342	3010	0.053	0.667	1990	0.520	0.205
	1480	0.000	2.152	1480	0.000	1.809	1480	0.000	1.379	1150	0.221	0.569	850	0.645	0.206
	709	0.000	2.176	709	0.000	1.851	702	0.008	1.426	472	0.408	0.512	390	0.734	0.215
	327	0.000	2.230	326	0.001	1.947	291	0.092	1.371	196	0.570	0.497	174	0.805	0.228
	157	0.000	2.397	151	0.028	2.067	115	0.290	1.204	88.4	0.690	0.503	81.9	0.858	0.242
	81.5	0.003	2.751	67.2	0.164	1.974	52.0	0.477	1.129	44.1	0.775	0.521	41.9	0.896	0.257
	37.8	0.078	3.139	28.0	0.400	1.833	24.0	0.644	1.141	21.8	0.846	0.560	21.1	0.929	0.281
	13.3	0.485	2.481	12.0	0.662	1.779	11.3	0.795	1.220	10.8	0.909	0.632	10.6	0.958	0.323
400	8080	0.000	2.021	8080	0.000	1.705	8080	0.000	1.308	8040	0.003	0.748	6020	0.266	0.285
	3030	0.000	1.975	3030	0.000	1.686	3030	0.000	1.327	2860	0.045	0.764	1900	0.499	0.264
	1350	0.000	1.900	1350	0.000	1.667	1350	0.000	1.373	1070	0.201	0.700	781	0.631	0.274

Table 38 (Continued)

	r = 0.0			r = 0.2			r = 0.3			r = 0.4			r = 0.45		
Fc	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu	Pe_L/z	T	ρu
400	678	0.000	1.817	678	0.000	1.661	672	0.007	1.448	462	0.375	0.653	76	0.717	0.287
	340	0.000	1.730	340	0.001	1.706	314	0.061	1.490	209	0.527	0.637	183	0.785	0.302
	160	0.000	1.727	156	0.022	1.848	122	0.245	1.364	91.3	0.662	0.630	83.9	0.845	0.312
	82.0	0.005	2.040	69.2	0.146	1.888	53.6	0.446	1.246	44.9	0.759	0.616	42.5	0.889	0.311
	37.2	0.094	2.682	28.3	0.390	1.801	24.3	0.632	1.191	22.0	0.840	0.609	21.3	0.926	0.311
	13.3	0.496	2.390	12.0	0.664	1.762	11.3	0.795	1.231	10.8	0.909	0.648	10.6	0.958	0.333
800	4970	0.000	1.794	4970	0.000	1.566	4970	0.000	1.282	4920	0.007	0.872	3510	0.332	0.379
	1760	0.000	1.570	1760	0.000	1.458	1760	0.000	1.319	1580	0.087	0.957	1070	0.543	0.392
	769	0.000	1.275	769	0.000	1.344	766	0.002	1.423	570	0.274	0.931	438	0.665	0.421
	361	0.000	0.915	361	0.000	1.271	347	0.030	1.604	233	0.452	0.918	197	0.750	0.448
	165	0.000	0.556	161	0.016	1.383	134	0.175	1.650	96.6	0.610	0.894	87.2	0.821	0.546
	81.6	0.020	0.808	72.2	0.121	1.648	56.6	0.388	1.472	46.3	0.726	0.811	43.4	0.874	0.423
	32.9	0.160	2.011	26.6	0.402	1.724	23.1	0.627	1.273	20.9	0.836	0.696	20.2	0.923	0.364
	12.6	0.539	2.221	11.6	0.683	1.729	11.0	0.804	1.251	10.5	0.913	0.678	10.3	0.959	0.354
-100	5610	0.000	2.350	5610	0.000	1.907	5610	0.000	1.356	5440	0.023	0.566	3550	0.481	0.120
	2600	0.000	2.485	2600	0.000	1.991	2600	0.000	1.374	2160	0.154	0.427	1520	0.620	0.105
	1330	0.000	2.646	1330	0.000	2.087	1330	0.001	1.383	918	0.361	0.322	740	0.711	1.100
	417	0.000	3.037	416	0.001	2.292	370	0.093	1.205	246	0.595	0.285	221	0.816	0.111
	175	0.000	3.444	169	0.026	2.380	123	0.343	0.958	97.3	0.717	0.317	91.0	0.866	0.138
	51.6	0.019	4.012	37.1	0.339	1.913	31.0	0.614	1.022	27.8	0.854	0.449	26.3	0.927	0.215
	15.8	0.399	2.772	13.8	0.614	1.825	12.8	0.771	1.187	12.1	0.902	0.588	11.8	0.958	0.295
-200	9630	0.000	2.341	9630	0.000	1.906	9630	0.000	1.356	9570	0.005	0.582	6570	0.376	1.112
	4350	0.000	2.503	4350	0.000	1.996	4350	0.000	1.370	4010	0.062	0.456	2620	0.563	0.080
	1680	0.000	2.772	1680	0.000	2.153	1680	0.001	1.378	1190	0.330	0.270	940	0.701	0.064
	752	0.000	3.082	752	0.000	2.311	732	0.021	1.305	466	0.517	0.217	404	0.783	0.064
	336	0.000	3.477	335	0.002	2.468	273	0.176	1.021	192	0.653	0.220	176	0.838	0.079
	104	0.000	4.118	88.5	0.136	2.208	66.0	0.484	0.887	56.2	0.780	0.313	53.4	0.903	0.137
	41.4	0.044	4.210	29.3	0.405	1.894	25.1	0.656	1.030	22.9	0.856	0.453	22.2	0.932	0.217
	12.8	0.488	2.637	11.5	0.674	1.807	10.8	0.801	1.205	10.4	0.915	0.605	10.2	0.957	0.306

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